Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2014.1232nm

Vol. **7**, No. 3, pp. 317-333
August 2014

# Simultaneous Approximation of Sobolev Classes by Piecewise Cubic Hermite Interpolation

Guiqiao Xu* and Zheng Zhang

*Department of Mathematics, Tianjin Normal University, Tianjin, 300387, P.R. China.*

Received 27 September 2012; Accepted (in revised version) 26 February 2014

Available online 12 August 2014

**Abstract.** For the approximation in $L_p$-norm, we determine the weakly asymptotic orders for the simultaneous approximation errors of Sobolev classes by piecewise cubic Hermite interpolation with equidistant knots. For $p = 1, \infty$, we obtain its values. By these results we know that for the Sobolev classes, the approximation errors by piecewise cubic Hermite interpolation are weakly equivalent to the corresponding infinite-dimensional Kolmogorov widths. At the same time, the approximation errors of derivatives are weakly equivalent to the corresponding infinite-dimensional Kolmogorov widths.

**AMS subject classifications**: 41A15, 65D07, 41A28

**Key words**: Piecewise cubic Hermite interpolation, $L_p$-norm, simultaneous approximation, equidistant knot, infinite-dimensional Kolmogorov width.

## 1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the set of all positive integers, all integers and all real numbers, respectively. For $1 \leq p \leq \infty$, let $L_p$ be the spaces of functions $f : \mathbb{R} \to \mathbb{R}$ with the corresponding norms $\|\cdot\|_p$. Denote by $W_p^r(\mathbb{R})$, $r \in \mathbb{N}$, the class of functions $f$ such that $f^{(r-1)}(f^{(0)} := f)$ is locally absolutely continuous and $\|f^{(r)}\|_p \leq 1$.

The approximation of periodic Sobolev classes by periodic polynomial splines with restrictions on its derivatives has been studied for a long time (see [1–4, 7–9]). In these researches, the approximation polynomial splines are assumed with equidistant knots and with defect 1. Recently, [14] and [15] consider the approximation of non-periodic Sobolev classes by polynomial splines with restrictions, where the approximation polynomial splines are also assumed with equidistant knots and with defect 1.

The simultaneous approximation problems for smooth functions are an important research topic in approximation theory and application. For polynomial simultaneous

*Corresponding author. *Email address:* `Xuguiqiao@eyou.com` (G. Q. Xu)

 

approximation problem, the main results can be found in [6] and [12]. As to the concrete interpolation polynomial operators, the main results can be looked up in [11] and [13]. As far as we know, all relevant results are only connected to a single function approximation and not for function classes approximation. In [1–4, 7–9, 14, 15], all results are connected to function approximation only, however the simultaneous approximation problems can also be discussed since both the functions and approximation splines have derivatives up to $r$-order. Hence, we want to consider the simultaneous approximation of Sobolev classes by piecewise cubic Hermite interpolation with equidistant knots. It is well known that the defect of piecewise cubic Hermite interpolation is 2.

Now we give the definition of piecewise cubic Hermite interpolation on knots $x_k = k/n$, $k \in \mathbb{Z}$. For $f \in C^{(1)}(\mathbb{R})$, there is an unique piecewise cubic polynomial $H_n(f,x)$ with knots $x_k = k/n$, $k \in \mathbb{Z}$ and satisfies the following conditions:

(1). $H_n(f,x) \in C^{(1)}(\mathbb{R})$;

(2). $H_n(f,x_k) = f(x_k)$, $H_n'(f,x_k) = f'(x_k)$, $k \in \mathbb{Z}$;

(3). $H_n(f,x)$ is a cubic polynomial about $x$ on each subinterval $[x_{k-1}, x_k]$, $k \in \mathbb{Z}$.

It is well known that for $x \in [x_k, x_{k+1}]$,

$$\begin{aligned} H_n(f,x) =& f(x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+\frac{2(x-x_k)}{x_{k+1}-x_k}\right) \\ &+ f'(x_k)(x-x_k)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2 \\ &+ f(x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2\left(1+\frac{2(x-x_{k+1})}{x_k-x_{k+1}}\right) \\ &+ f'(x_{k+1})(x-x_{k+1})\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2. \end{aligned} \tag{1.1}$$

On the one hand, we will consider the second derivative approximation of piecewise cubic Hermite interpolation on Sobolev classes $W_p^3(\mathbb{R})$. We obtain the following results.

**Theorem 1.1.** *Let $H_n(f,x)$ be defined as (1.1). Then we have*

$$\sup_{f\in W_\infty^3(\mathbb{R})} \|H_n'' f - f''\|_\infty = \frac{8}{27n}, \tag{1.2}$$

$$\sup_{f\in W_1^3(\mathbb{R})} \|H_n'' f - f''\|_1 = \frac{C_1}{n}, \tag{1.3}$$

*and for $1 < p < \infty$,*

$$\sup_{f\in W_p^3(\mathbb{R})} \|H_n'' f - f''\|_p \le \left(\frac{8}{27}\right)^{1-\frac{1}{p}} C_1^{\frac{1}{p}} \frac{1}{n}. \tag{1.4}$$

*Where $C_1$ is defined in (3.30).*