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Legendre-Gauss Spectral Collocation Method for Second Order Nonlinear Delay Differential Equations

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Abstract. In this paper, we present and analyze a single interval Legendre-Gauss spectral collocation method for solving the second order nonlinear delay differential equations with variable delays. We also propose a novel algorithm for the single interval scheme and apply it to the multiple interval scheme for more efficient implementation. Numerical examples are provided to illustrate the high accuracy of the proposed methods.

AMS subject classifications: 65M70, 65L05, 65L70, 41A10

Key words: Legendre-Gauss spectral collocation method, second order nonlinear delay differential equations, error analysis.

1. Introduction

Delay differential equations (DDEs) constitute basic mathematical models for real phenomena, for instance in engineering, chemical process, economics and biological systems. Over the past few decades, rapid progress has been made in numerical methods for various DDEs, see, for example, [2, 3, 6, 33] for an overview. Many numerical schemes mainly based on the Taylor's expansions or quadrature formulas introduced for initial value problems of ordinary differential equations (ODEs) have also been frequently used for numerical solutions of DDEs (cf. [7, 16, 17, 20]).

As we know, spectral methods are widely used in numerical solutions of partial differential equations (cf. [4, 5, 8, 10–12, 24, 25]), which have become powerful tools for solving many kinds of differential equations arising in various fields of engineering and science. Among many types of spectral methods that are more applicable and

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frequently used are collocation methods. In recent years, spectral collocation methods have become increasingly popular in numerical solutions of initial value problems of ODEs and DDEs due to their high-order accuracy and easy implementation. For instance, Guo et al. developed several Legendre-Gauss-type spectral collocation methods for initial value problems of ODEs (cf. [13–15,26]); Kanyamee and Zhang [19] investigated the Legendre/Chebyshev-Gauss-Lobatto spectral collocation method for solving Hamiltonian dynamical systems; Ito et al. [18] proposed a Legendre-tau method for linear DDEs with one constant delay; Ali et al. [1] developed a Legendre collocation method for linear DDEs with vanishing proportional delays; Wei and Chen [28, 29] studied Legendre collocation methods for linear Volterra integro-differential equations and linear Volterra integro-differential equations with pantograph delay. Actually, due to the nature of the DDEs and the behavior of the solutions, it is a difficult task to design efficient codes for the numerical solutions of DDEs, particularly, for the nonlinear DDEs. Very recently, we note that Wang et al. [27, 30] presented Legendre-Gauss-type spectral collocation methods for solving first order nonlinear DDEs. However, to the best of our knowledge, there are few discussions on the numerical methods for second order nonlinear DDEs.

The aim of this paper is to develop a Legendre-Gauss spectral collocation method for solving the second order nonlinear DDE with variable delay:

$$\begin{cases} U''(t) = f(U(t), U'(t), V(t), W(t), t), & 0 < t \leq T, \\ U(t) = \varphi(t), \quad U'(t) = \varphi'(t), & t \leq 0, \end{cases} \tag{1.1}$$

where $V(t) = U(t - \theta(t)), W(t) = U'(t - \theta(t))$, f, φ are given functions and the delay variable $\theta(t) \geq 0$.

We first propose a single interval Legendre-Gauss spectral collocation scheme for problem (1.1) motivated by [15, 27], and design a novel algorithm by full utilizing properties of the Legendre polynomials. Roughly speaking, we expand the numerical solution by a truncated shifted Legendre polynomial series, and collocate the numerical scheme at the Legendre-Gauss points to determine the expansion coefficients. For more efficient implementation, we also introduce a multiple interval Legendre-Gauss spectral collocation scheme. These approaches we present here have several striking features:

- The single interval Legendre-Gauss collocation scheme can be implemented easily and efficiently for nonlinear problems due to the proposed novel algorithm (see Subsection 2.2).

- The multiple interval Legendre-Gauss collocation scheme enable us to solve the resultant system efficiently and economically. Specifically, if T is large, we can divide the solution interval $(0, T)$ into subintervals and solve the subsystems successively. Moreover, the resultant system for the expansion coefficients of the numerical solution with a modest number of unknowns can be solved quickly.

- In actual computation, we only need to store the expansion coefficients of the