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Extremal Eigenvalues of the Sturm-Liouville Problems with Discontinuous Coefficients

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Abstract. In this paper, an extremal eigenvalue problem to the Sturm-Liouville equations with discontinuous coefficients and volume constraint is investigated. Liouville transformation is applied to change the problem into an equivalent minimization problem. Finite element method is proposed and the convergence for the finite element solution is established. A monotonic decreasing algorithm is presented to solve the extremal eigenvalue problem. A global convergence for the algorithm in the continuous case is proved. A few numerical results are given to depict the efficiency of the method.

AMS subject classifications: 65L60, 65L15, 34B09

Key words: Extremal eigenvalue problem, Sturm-Liouville problem, finite element method, convergence analysis.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$ $(n=1,2,3)$, consider the following eigenvalue problem:

$$\begin{cases} -\mathrm{div}(\sigma(x)\nabla u)=\lambda u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\sigma(x)$ is a positive piecewise constant function. For any given such function σ, it is known (cf. [11,16,22]) that the Eq. (1.1) admits a sequence of eigenvalues

$$0<\lambda_1\le\lambda_2\le\cdots\to\infty,$$

and its smallest eigenvalue λ_1 is denoted by $\lambda_1(\sigma)$. We are interested in the minimization of the first eigenvalue $\lambda_1(\sigma)$ among all possible choices of function $\sigma(x)$. This extremal eigenvalue problem arises from a lot of structural engineering and optimal design problems (cf. [1,5]). For example, if we consider the non-homogenous heat conductor with different

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conductivities, the first eigenvalue and its corresponding eigenfunction represents the first mode of heat diffusion pattern. In particular, we assume that the heat conductor is made by two materials with conductivities α and β, $0 < \alpha < \beta$. Let the materials with the conductivity α occupies a measurable set $D \subset \Omega$, then

$$\sigma(x) = \alpha\chi_D + \beta\chi_{\Omega\backslash D},$$

where χ_D is the characteristic function of D. The material with conductivity α are assumed to have a fixed volume, which leads to the following optimization problem:

Problem 1.1. For

$$\min_{\sigma\in\mathscr{A}} \lambda_1(\sigma),$$

where $\mathscr{A}$ is the admissible set for all possible choices of the conductivity function which is defined as:

$$\mathscr{A} = \left\{\sigma : \sigma = \alpha\chi_D + \beta\chi_{\Omega\backslash D},\ \fint_\Omega \sigma = c\right\},$$

where $\fint$ is the average of integral function on the domain and c is a constant which satisfies: $\alpha \le c \le \beta$.

The condition to the existence for the minimizer of this problem remains an open question. From the work of Murat and Tartar on a control problem involving immiscible fluids [26], it is known that the Problem 1.1 may not always possess a solution, and in general one should consider the framework of homogenization theory. Existence of a solution and optimality conditions in the class of relaxed designs has been discussed in Cox and Lipton [12].

If the domain Ω is an interval in $\mathbb{R}^1$ or a ball in $\mathbb{R}^n$, the existence of the minimizer has been studied in various papers. The one dimensional problem was solved by Krein [23] by exploiting the equivalence between this problem and a similar extremal eigenvalue problem for a composite membrane with variable densities. The technique is so-called Liouville transformation, to transfer the variable conductivity σ into the lower-order term, then one can use the results of extremal eigenvalue problem for a composite membrane, see Cox, Mclaughlin [14, 15]. The Liouville transformation can be found in many papers in the context of Sturm-Liouville problems with discontinuous coefficients such as [6, 18, 19, 25]. When the domain is a ball, the existence of a radially symmetric minimizer has been proved in [2] by using rearrangement technique.

On the other hand, the numerical treatment to the extremal eigenvalue problem of a variable density membrane have been studied in [13–15, 17, 29], but there are only few works for the extremal eigenvalue problem with variable conductivity (cf. [9, 10, 12]). The finite element method for the eigenvalue problem have been studied extensively, see [1, 7, 8, 30] for constant conductivity function case and [3] for discontinuous conductivity. The computational result can also be found in Nemat-Nasser et al. [27,28]. Recently, Liang et al. [24] study the convergence of the finite element method for the extremal eigenvalue problem with variable density function. Inspired by the previous works, we exploit