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Laguerre Spectral Method for High Order Problems

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Abstract. In this paper, we propose the Laguerre spectral method for high order problems with mixed inhomogeneous boundary conditions. It is also available for approximated solutions growing fast at infinity. The spectral accuracy is proved. Numerical results demonstrate its high effectiveness.

AMS subject classifications: 65L60, 65M70

Key words: Laguerre spectral method, high order problems with mixed inhomogeneous boundary conditions.

1. Introduction

The spectral method possesses high accuracy, and so plays an important role in numerical solutions of differential and integral equations, see [2–4, 7–9, 18] and the references therein. During the past two decades, more and more attentions were paid to problems defined on various unbounded domains. The Laguerre spectral method has been used widely for differential equations defined on the half line and the related unbounded domains, as well as certain exterior problems, see [10–12,14,15,17,19,20] and the references therein. But, there are still two unsettled problems. Firstly, in the existing work, we usually reformed original problems by some variable transformations, and then solved the alternative formulations by using the Laguerre approximation. Thus, those spectral schemes seem available essentially for approximated solutions decaying to zero at infinity. However, in many cases, the solutions do not tend to zero at infinity, such as the kink-like solitons, the heteroclinic solutions in biology, the solutions of Harry-Dym equation and some nonlinear

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dynamical systems. Next, since the existing results on the Laguerre approximation are not optimal, the error estimates of numerical solutions are not very precise. On the other hand, we considered second order problems mostly. Whereas, in some practical cases, such as the stream function form of the Navier-Stokes equations, we have to deal with high order problems. Recently, Guo, Sun and Zhang [13] proposed the generalized Laguerre quasi-orthogonal approximation, which leads to the probability of producing new Laguerre spectral method suitable also for high order problems with solutions growing fast at infinity, and deriving better error estimates of numerical solutions. We refer to the review paper of Guo, Zhang and Sun [16].

In this paper, we investigate the new Laguerre spectral method for high order problems with mixed inhomogeneous boundary conditions. The next section is for preliminaries. In Section 3, we consider two fourth order problems with various boundary conditions. We design the spectral schemes and prove their spectral accuracy. They are also suitable for high order problems with solutions growing fast at infinity. Moreover, we provide the Laguerre spectral method with exact imposition of boundary conditions. In Section 4, we present some numerical results demonstrating the high effectiveness of suggested algorithms. The final section is for concluding remarks.

2. Preliminaries

Let $\Lambda = \{x|0 < x < \infty\}$ and $\chi(x)$ be certain a weight function. For integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(\cdot,\cdot)_{r,\chi,\Lambda}$, the semi-norm $|\cdot|_{r,\chi,\Lambda}$ and the norm $\|\cdot\|_{r,\chi,\Lambda}$. In particular, the inner product and the norm of $L^2_\chi(\Lambda)$ are denoted by $(\cdot,\cdot)_{\chi,\Lambda}$ and $\|\cdot\|_{\chi,\Lambda}$, respectively. For simplicity, we denote d^kv/dx^k by $\partial^k_x v$. For integer $r \geq 1$,

$$_0H^r_\chi(\Lambda) = \{v \in H^r_\chi(\Lambda) | \partial^k_x v(0) = 0,\ 0 \leq k \leq r-1\}.$$

We omit the subscript χ in notations whenever $\chi(x) \equiv 1$.

The scaled generalized Laguerre polynomials of degree $l \geq 0$ were given in [17], as

$$L^{(\alpha,\beta)}_l(x) = \frac{1}{l!} x^{-\alpha} e^{\beta x} \partial^l_x (x^{l+\alpha} e^{-\beta x}), \quad \alpha > -1, \quad \beta > 0, \quad l \geq 0.$$

In this work, we shall use the specific base functions $\mathscr{L}^{(-m,\beta)}_l(x)$ with integer $m \geq 1$, namely (see [13]),

$$\mathscr{L}^{(-m,\beta)}_l(x) = x^m L^{(m,\beta)}_{l-m}(x), \quad l \geq m. \tag{2.1}$$

Let $\omega_{-m,\beta}(x) = x^{-m} e^{-\beta x}$. By (4.8) of [13], we have

$$\int_\Lambda \mathscr{L}^{(-m,\beta)}_l(x) \mathscr{L}^{(-m,\beta)}_{l'}(x) \omega_{-m,\beta}(x) dx = \eta^{(-m,\beta)}_l \delta_{l,l'}, \tag{2.2}$$