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Superconvergence and L^∞-Error Estimates of the Lowest Order Mixed Methods for Distributed Optimal Control Problems Governed by Semilinear Elliptic Equations

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Abstract. In this paper, we investigate the superconvergence property and the L^∞-error estimates of mixed finite element methods for a semilinear elliptic control problem. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control variable is approximated by piecewise constant functions. We derive some superconvergence results for the control variable. Moreover, we derive L^∞-error estimates both for the control variable and the state variables. Finally, a numerical example is given to demonstrate the theoretical results.

AMS subject classifications: 49J20, 65N30

Key words: Semilinear elliptic equations, distributed optimal control problems, superconvergence, L^∞-error estimates, mixed finite element methods.

1. Introduction

It is well known that the finite element approximation plays an important role in the numerical treatment of optimal control problems. There have been extensive studies in convergence and superconvergence of finite element approximations for optimal control problems, see, for example, [1,6,11–13,19–23]. A systematic introduction of finite element methods for PDEs and optimal control problems can be found in, for example, [9,16].

In many control problems, the objective functional contains the gradient of the state variable. Thus, the accuracy of the gradient is important in the numerical approximation of the state equations. In the finite element community, mixed finite element methods are optimal for discretization of the state equations in such cases, since both the scalar variable and its flux variable can be approximated in the same accuracy by using mixed finite element methods. For a priori error estimates and superconvergence properties of mixed

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finite elements for optimal control problems, see, for example, [4,5,8]. In [5], Chen used the postprocessing projection operator, which was defined by Meyer and Rösch (see [19]) to prove a quadratic superconvergence of the control by mixed finite element methods. Recently, Chen et al. derived error estimates and superconvergence of mixed methods for convex optimal control problems in [8]. However, as far as we know there is no superconvergence analysis in mixed finite element methods for optimal control problems governed by semilinear elliptic equations except [7].

The goal of this paper is to derive the superconvergence property and the L^∞-error estimates of mixed finite element approximation for a semilinear elliptic control problem. Firstly, we derive the superconvergence property between average L^2 projection and the approximation of the control variable, the convergence order is $h^{3/2}$ as that obtained in [8]. Then, a global superconvergence result for the control variable can be obtained by using a recovery operator. We also derive the L^∞-error estimates for both the control variable and the state variables. Finally, we present a numerical experiment to demonstrate the practical side of the theoretical results about superconvergence and L^∞-error estimates.

We consider the following semilinear optimal control problems for the state variables $\boldsymbol{p}$, y, and the control u with pointwise constraint:

$$\min_{u\in U_{ad}} \left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\mathrm{div}(A(x)\mathbf{grad}y)+\phi(y)=u, \qquad x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\mathrm{div}\boldsymbol{p}+\phi(y)=u, \quad \boldsymbol{p}=-A(x)\mathbf{grad}y, \quad x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0, \quad x\in\partial\Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$. U_{ad} denotes the admissible set of the control variable, defined by

$$U_{ad}=\{u\in L^\infty(\Omega):\ u\geq 0,\ \ a.e.\ \text{ in }\Omega\}. \tag{1.5}$$

We assume that the function $\phi(\cdot)\in W^{2,\infty}(-R,R)\cap H^3(-R,R)$ for any $R>0$, $\phi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\phi'\geq 0$. Moreover, we assume that $y_d\in W^{1,\infty}(\Omega)$ and $\boldsymbol{p}_d\in(H^2(\Omega))^2$. ν is a fixed positive number. The coefficient $A(x)=(a_{ij}(x))$ is a symmetric matrix function with $a_{ij}(x)\in W^{1,\infty}(\Omega)$, which satisfies the ellipticity condition

$$c_*|\xi|^2\leq\sum_{i,j=1}^{2}a_{ij}(x)\xi_i\xi_j, \quad \forall(\xi,x)\in\mathbb{R}^2\times\bar{\Omega}, \ \ c_*>0.$$