Numer. Math. Theor. Meth. Appl. Vol. **6**, No. 1, pp. 262-275
doi: 10.4208/nmtma.2013.mssvm14 February 2013

Iterative Methods of Richardson-Lucy-Type for Image Deblurring

M. K. Khan[1], S. Morigi[2], L. Reichel[1] and F. Sgallari[3,*]

[1] *Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA.*
[2] *Department of Mathematics, University of Bologna, P.zza Porta San Donato 5, Bologna, Italy.*
[3] *Department of Mathematics–CIRAM, University of Bologna, Via Saragozza 8, Bologna, Italy.*

Received 18 December 2011; Accepted (in revised version) 11 September 2012

Available online 11 January 2013

Abstract. Image deconvolution problems with a symmetric point-spread function arise in many areas of science and engineering. These problems often are solved by the Richardson-Lucy method, a nonlinear iterative method. We first show a convergence result for the Richardson-Lucy method. The proof sheds light on why the method may converge slowly. Subsequently, we describe an iterative active set method that imposes the same constraints on the computed solution as the Richardson-Lucy method. Computed examples show the latter method to yield better restorations than the Richardson-Lucy method and typically require less computational effort.

AMS subject classifications: 65R20, 65R32, 65K05

Key words: Constrained ill-posed problem, nonnegativity, active set method, image restoration.

1. Introduction

This paper is concerned with the restoration of images that have been contaminated by blur and noise. We consider two-dimensional gray-scale images, whose brightness is represented by a real-valued function defined on a square or rectangular region $\Omega \in \mathbb{R}^2$. Let the function b^δ represent the available observed blur- and noise-contaminated image, and let the function $\hat{x}$ represent the unknown associated blur- and noise-free image that we would like to recover. We assume the functions b^δ and $\hat{x}$ to be related by the degradation model

$$b^\delta(s) = \int_\Omega h(s,t)\hat{x}(t)dt + \eta^\delta(s), \qquad s \in \Omega, \tag{1.1}$$

*Corresponding author. *Email addresses:* `kazim@math.kent.edu` (M. K. Khan), `serena.morigi@unibo.it` (S. Morigi), `reichel@math.kent.edu` (L. Reichel), `fiorella.sgallari@unibo.it` (F. Sgallari)

where η^δ represents additive noise (error) in the data b^δ. The kernel h models the blurring and is often referred to as the point-spread function (PSF). In many applications, the integral is a symmetric convolution, i.e., the kernel h is of the form

$$h(s,t) = h(t,s) = k(s-t), \qquad s,t \in \Omega, \tag{1.2}$$

for some function k. This situation is of primary interest to us; however, the method described in Section 3 can be applied to restoration problems with a more general kernel. We also will assume that

$$\int_\Omega h(s,t)ds = 1, \qquad t \in \Omega. \tag{1.3}$$

Then the blurring does not change the total brightness of the image.

The PSF in typical image restoration problems is smooth or piecewise smooth with jump discontinuities. The integral operator (1.1) then is compact and therefore its singular values cluster at the origin. Consequently, the integral operator does not have a bounded inverse.

We would like to determine an accurate approximation of the unknown blur- and noise-free image $\hat{x}$ when the observed image b^δ and the kernel h are available. Straightforward solution of

$$\int_\Omega h(s,t)x(t)dt = b^\delta(s), \qquad s \in \Omega, \tag{1.4}$$

for x generally does not yield a useful approximation of the desired blur- and noise-free image $\hat{x}$ because of the noise η^δ in b^δ and the fact that the integral operator does not have a bounded inverse. Due to the latter, the task of solving (1.4) is an ill-posed problem; see, e.g., Engl et al. [4] for discussions on ill-posed problems and on numerical methods for their solution.

We seek to determine an accurate approximation of $\hat{x}$ by computing a suitable approximate solution of (1.4). It is generally beneficial to impose constraints known to be satisfied by $\hat{x}$ on the computed approximation during the solution process. Since $\hat{x}$ represents the brightness of the image, it is nonnegative. We would like the computed approximation of $\hat{x}$ to satisfy the same constraint, i.e., would like our solution method to determine an approximate solution x of (1.4) that satisfies

$$x(t) \geq 0, \qquad t \in \Omega. \tag{1.5}$$

Integrating (1.4) with respect to s and using (1.3) yields

$$\int_\Omega x(t)dt = \int_\Omega b^\delta(s)ds. \tag{1.6}$$

We also would like the computed approximate solution to satisfy this constraint.

The present paper discusses two methods for determining approximate solutions of (1.4) that satisfy the constraints (1.5) and (1.6). The first method considered is the classical Richardson-Lucy method introduced by Richardson [15] and Lucy [8]. This method