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The Simultaneous Approximation Average Errors for Bernstein Operators on the r-Fold Integrated Wiener Space

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Abstract. For weighted approximation in L_p-norm, we determine strongly asymptotic orders for the average errors of both function approximation and derivative approximation by the Bernstein operators sequence on the r-fold integrated Wiener space.

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Key words: Bernstein operators, weighted L_p-norm, r-fold integrated Wiener space, average error.

1. Introduction

Let F be a real separable Banach space equipped with a probability measure μ on the Borel field of F. Let H be another normed space with norm $\|\cdot\|$ such that F is continuously embedded in H. Any $A : F \to H$ such that $f \mapsto \|f - A(f)\|$ is a measurable mapping is called an approximation operator (or just approximation). The p-average error of A is defined as

$$e_p(A, \|\cdot\|, F, \mu) = \left(\int_F \|f - A(f)\|^p \mu(df) \right)^{\frac{1}{p}}.$$

Since in practice the underlying function is usually given via its values at finitely many points, the approximation operator $A(f)$ is often considered depending on some function values about f only. Many papers such as [1-4] studied the complexity of computing an ε-approximation in the average case setting. We observe that all results used before rely on spline function approximation (see [5]). It is well known that the optimal spline function approximation operators depend on the smoothness property of covariance kernels, and it is not known if they are good approximation operators for derivative approximation. The simultaneous approximation problem for smooth functions is an important research topic in approximation theory and application. We want to consider the simultaneous approximation problem in the average case setting. Since the Bernstein operator approximation

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is the most important approximation method which depends on its values at equidistant interpolation points only, and it is widely used in practice, we will discuss its simultaneous approximation average errors on the r-fold integrated Wiener space.

Denote

$$F_0 = \{f \in C[0,1] : f(0) = 0\},$$

and for every $f \in F_0$ set

$$\|f\|_C := \max_{0 \le t \le 1} |f(t)|.$$

Then $(F_0, \|\cdot\|_C)$ becomes a separable Banach space. Denote by $\mathfrak{B}(F_0)$ the Borel field of $(F_0, \|\cdot\|_C)$, and by ω_0 the Wiener measure on $\mathfrak{B}(F_0)$ (see [5]). Now we introduce integral operator $T_r, r \ge 1$, on F_0 as follows.

Let $r \ge 0$ be an integer. Define for $r = 0$

$$(T_0 g)(t) = g(t), \ \ \forall g \in F_0,$$

and for $r \ge 1$

$$(T_r g)(t) = \int_0^t g(u) \cdot \frac{(t-u)^{r-1}}{(r-1)!} du.$$

Obviously we have

$$(T_r g)^{(s)}(t) = (T_{r-s} g)(t), \ \ \forall 0 \le s \le r, \tag{1.1}$$

and for an arbitrary $g \in F_0$,

$$T_r g \in F_r = \left\{ f \in C^{(r)}[0,1] : f^{(k)}(0) = 0 \text{ for } k = 0, 1, \cdots, r \right\}.$$

It is well known that T_r is a bijective mapping from F_0 to F_r. The r-fold integrated Wiener measure ω_r on F_r is defined by induced measure $\omega_r = T_r \omega_0$, i.e., for $A \subset F_r$,

$$\omega_r(A) = \omega_0(\{g \in F_0 | T_r g \in A\}).$$

From [5, p. 70] and [5, p. 71] we know

$$\int_{F_0} f(s) f(t) \omega_0(df) = \min\{s, t\}, \ \ \forall s, t \in [0,1], \tag{1.2}$$

$$\int_{F_r} f(s) f(t) \omega_r(df) = \int_0^1 \frac{(s-u)_+^r (t-u)_+^r \, du}{(r!)^2}. \tag{1.3}$$

Where $z_+ = z$ if $z > 0$ and $z_+ = 0$ otherwise.

For $\varrho \in L_1[0,1]$, $\varrho \ge 0$, the weighted L_p-norm of $f \in C[0,1]$ is defined by

$$\|f\|_{p,\varrho} = \left(\int_0^1 |f(t)|^p \cdot \varrho(t) dt \right)^{\frac{1}{p}}$$