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Second Order Multigrid Methods for Elliptic Problems with Discontinuous Coefficients on an Arbitrary Interface, I: One Dimensional Problems

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Abstract. In this paper we present a one dimensional second order accurate method to solve Elliptic equations with discontinuous coefficients on an arbitrary interface. Second order accuracy for the first derivative is obtained as well. The method is based on the Ghost Fluid Method, making use of ghost points on which the value is defined by suitable interface conditions. The multi-domain formulation is adopted, where the problem is split in two sub-problems and interface conditions will be enforced to close the problem. Interface conditions are relaxed together with the internal equations (following the approach proposed in [10] in the case of smooth coefficients), leading to an iterative method on all the set of grid values (inside points and ghost points). A multigrid approach with a suitable definition of the restriction operator is provided. The restriction of the defect is performed separately for both sub-problems, providing a convergence factor close to the one measured in the case of smooth coefficient and independent on the magnitude of the jump in the coefficient. Numerical tests will confirm the second order accuracy. Although the method is proposed in one dimension, the extension in higher dimension is currently underway [12] and it will be carried out by combining the discretization of [10] with the multigrid approach of [11] for Elliptic problems with non-eliminated boundary conditions in arbitrary domain.

AMS subject classifications: 35J25, 65N06, 65N55

Key words: Elliptic equation, discontinuous coefficient, second order accuracy, multigrid, arbitrary interface, jump conditions.

Introduction

Elliptic equations with jumping coefficients across a one-codimensional interface Γ arise in several applications. Let us mention as examples the steady-state diffusion problem in two materials with different diffusion coefficient separated by an arbitrary interface,

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the Poisson equation coming from the projection method in incompressible Navier-Stokes equation for fluids with different density, the porous-media equation to model the oil reservoir, electrostatic problems, and many others. In order to close the problem, interface conditions related to the jump of the solution and of the flux across the interface are included. In all these problems the interface may be arbitrary (not aligned with a line grid) and can change in time.

Numerous techniques have been developed to treat such problem. Interface-fitted grid methods such as the ones based on Finite Elements Methods [3, 5] are not suitable in case of moving interface, because a re-meshing grid is needed at each time step and this makes the computation expensive. Then an approach treating the interface embedded in a Cartesian grid and moving according to the velocity field of the fluid is preferred. Since the interface may not be aligned with the grid, a special treatment is needed. The simplest method makes use of the Shortley-Weller discretization [30], that discretizes the Laplacian operator with usual central difference away from the interface, and makes use of a non symmetric stencil in the points close to the interface, adding extra-grid points on Γ. While jumping condition on the solution is straightforward to discretize on interface points, the jump in the flux (involving the normal derivative) cannot be immediatly discretized in more than one dimension. In fact, Shortley-Weller discretization requires that the value of the normal derivative of the solution on both sides of the interface is suitably reconstructed at the intersection between the grid and the interface. This approach is adopted, for example, by Hackbusch in [20] to first order accuracy, and by other authors (see [6] and references therein) to second order accuracy. However, the method proposed by Bramble in [6] for second order accuracy is quite involved and not recommendable for practical purposes.

Methods based on embedding the domain in a Cartesian grid without adding extra-grid points are derived from the pioneering work of Peskin [26], where the Immersed Boundary Methods is introduced to model blood flows in the heart. In that paper a source term is localized on the the boundary and the method makes use of a discretized delta-function, leading to a first order accuracy. A second order accurate extension to jump coefficients is the Immersed Interface Methods, first developed by LeVeque and Li in [23]. Such method uses a six-point stencil to discretize the elliptic equation in grid points close to the interface Γ and the coefficients of such stencil are found by Taylor expansion of the solution. Jump conditions on the interface are then used to modify the coefficients appearing in the equation corresponding to nodes near Γ, in such a way that the overall discretization is second order accurate. Non-homogeneous jump conditions are allowed on the function and on the normal flux.

Another method which achieves second order accuracy by modifying standard difference formulas was proposed by Mayo in [24] for solving Poisson or biharmonic equation on irregular domains. Such method embeds the irregular domain in a regular region with a Cartesian grid and discretizes the equation on the whole region, by suitable extension of the solution outside.

In all these methods the only unknowns are the values on the grid points and the stencil may cross the interface, leading to a quite involved procedure to reach the desired