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A Triangular Spectral Method for the Stokes Equations

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Abstract. A triangular spectral method for the Stokes equations is developed in this paper. The main contributions are two-fold: First of all, a spectral method using the rational approximation is constructed and analyzed for the Stokes equations in a triangular domain. The existence and uniqueness of the solution, together with an error estimate for the velocity, are proved. Secondly, a nodal basis is constructed for the efficient implementation of the method. These new basis functions enjoy the fully tensorial product property as in a tensor-produce domain. The new triangular spectral method makes it easy to treat more complex geometries in the classical spectral-element framework, allowing us to use arbitrary triangular and tetrahedral elements.

AMS subject classifications: 65N35, 65N22, 65F05, 35J05

Key words: Stokes equations, triangular spectral method, error analysis.

1. Introduction

The spectral-element method is a high-order variational method which combines the geometric flexibility of finite-elements with the high accuracy of spectral methods. It exhibits several favorable computational properties, such as the use of tensor products, naturally diagonal mass matrices, and suitability for parallel computation. However, in order to use fast tensor product summation, the standard spectral-element method is usually restricted to quadrilateral partitions, which are difficult to use for adaptive computation in complex geometries. One way to overcome this drawback is to allow triangular partitions, which are more flexible in handling complex geometries.

Existing spectral methods on triangle can be classified into three categories according to the class of functions used in the approximations: (i) approximations by polynomials in triangle through mapping (see, e.g., [3,5,8,12,16]); (ii) approximations by polynomials in triangle using special nodal points such as Fekete points (see, e.g., [7,13,14,17]); and (iii)

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approximations by non-polynomial functions in triangle (see, e.g., [2,6,15]). The triangular spectral methods based on polynomials were motivated by the classical result that any smooth function can be well approximated by polynomials. The orthogonal polynomials on triangle, often referred as Dubiner's basis in the spectral-element community, were constructed by Koornwinder [9] and by Dubiner [5], and implemented in the spectral-element package NekTar (cf. [8,16]). A drawback of this approach is that no suitable interpolation operator, i.e., no corresponding nodal basis, is available for the Dubiner's basis, which involves a wrapped product, instead of the tensor product, of the Jacobi polynomials used to define the basis functions. The lack of nodal basis makes it difficult to implement in a collocation framework. Recently, a triangular spectral method using rational polynomials was proposed and analyzed (cf. [15]). The rational basis functions are constructed through the Duffy mapping in the reference square, and are mutually orthogonal with respect to a suitable weight function. In particular, one can construct a nodal basis through the Duffy mapping which allows simple and efficient implementation as a usual nodal spectral-element method.

In this paper, we will consider a triangular spectral method based on the rational function approximation for the Stokes equations. First, we construct and analyze a new triangular spectral method for the Stokes equations. By using a compatible pair of spaces for the velocity and pressure, we establish the well-posedness of the discrete problem, and derive an error estimate for the velocity. Although no theoretical estimate is provided for the inf-sup constant, numerical tests are carried out to investigate its asymptotic behavior. Second, we introduce a nodal basis, the transformation of which is the tensor product of the standard Lagrangian polynomials defined on the rectangular Gauss-Lobatto points, with an exception corresponding to the "collapsed" side. The remarkable advantage of this basis is that it enjoys the fully tensorial-product property, and easy to implement in case of multi-elements. The availability of nodal triangular basis greatly enhances the geometrical flexibility of spectral-element method by allowing fully unstructured mesh.

The paper is organized as follows. In the next section, we present some preliminary results which will be used in the sequel. Then we construct the triangular spectral method for the Stokes equations, and study its well-posedness and error estimate in Section 3. In Section 4, we provide implementation details, with a particular attention to the construction of the nodal basis functions. Some numerical results and discussions are presented in Section 5.

2. Construction of the triangular spectral method

Throughout this paper, we will use boldface letters to denote vectors and vector functions. Let c stand for a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ to mean that $A \leqslant cB$, and use the expression $A \cong B$ to mean that $A \lesssim B \lesssim A$. For a bounded domain Ω and a generic positive weight function ω, we denote the inner product of $L^2_\omega(\Omega)$ by

$$(u,v)_{\omega,\Omega} := \int_\Omega uv\omega d\mathbf{x}$$