Numer. Math. Theor. Meth. Appl. Vol. **11**, No. 4, pp. 782-794
doi: 10.4208/nmtma.2018.s06 November 2018

A Fourth Order Compact Scheme for Transport Equation with Discontinuous Coefficients

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Received 19 August 2017; Accepted (in revised version) 19 December 2017

Dedicated to Professor Xiaoqing Jin on the occasion of his 60th birthday

Abstract. This paper reports a compact fourth order accurate cubic spline collocation procedure for numerical solution of transport equation with discontinuous coefficients. The proposed scheme is shown to be stable. A new patch up technique is developed near the interface where the coefficient has jump discontinuity. A numerical experiment is given to confirm the accuracy and efficiency of the proposed scheme.

AMS subject classifications: 65M06

Key words: Compact scheme, cubic spline collocation procedure, transport equation, jump discontinuity, discontinuous coefficients.

1. Introduction

Consider the transport equation in one dimension

$$u_t = c(x)u_x, \qquad t>0, \quad x\in[a,b], \tag{1.1a}$$
$$u(x,0)=u_0(x), \quad x\in[a,b]. \tag{1.1b}$$

Here c is piecewise smooth having discontinuities across some interfaces within the solution domain. Transport equations with discontinuous coefficients arise in conservation laws. Recently this problem has created a lot of interest among researchers. The primary difficulty in using high order schemes is finding stable scheme at the interface. In recent years, many contributions [1–5] have been made to solve interface problems. Wiegmann

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and Bube [6] proposed the explicit jump immersed interface method for pde's with piecewise smooth functions. Zhou et al. [7] proposed the matched interface and boundary method for elliptic problems with interfaces. Chen and Strain [8] proposed a new Krylov-accelerated multi grid method with piecewise-polynomial discretization for elliptic interface problems. Ito et al. [9] have discussed immersed interface CIP method to solve one dimensional hyperbolic equations. Mohanty et al. [10] have derived fourth order Numerov type discretization to solve wave equation with variable coefficients. Collocation procedure for boundary value problems based on piecewise polynomial functions has been studied by many researchers [11,12]. Gardner [13] used cubic B-splines to solve regularized long wave equations.

In this paper we offer a high order compact scheme based on cubic B-spline collocation method to solve transport equation in one space dimension. We first present the scheme when the coefficient c is constant and extend it to the case when c is discontinuous. A new patch up technique is used to obtain a scheme at the interface. The outline of the paper is as follows: In Section 2, the cubic B-spline collocation method for solving transport equation with constant coefficient is described. Stability of the compact scheme is discussed in Section 3. In Section 4, the scheme is extended to the case when c is variable. In case of discontinuous coefficient, the scheme at interface is obtained by patching of two surfaces in Section 5. Stability for the scheme at interface is obtained in Section 6. A numerical experiment is performed in Section 7 to demonstrate the fourth order accuracy of the scheme.

2. Cubic spline collocation method

We begin by considering the problem (1.1a) with coefficient c to be constant. Let $\{a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b\}$ be a partition of the solution domain $[a, b]$ with the uniform spacing $h = x_i - x_{i-1} = (b-a)/N$ for $i = 1, 2, \cdots, N$.

Then s is cubic spline if $s \in S_3([a, b])$ where

$$S_3([a,b]) = \left\{ s \in C^2([a,b]) \,|\, s_i \in P_3, \quad \forall i = 1, 2, \cdots, N \right\}.$$

Here s_i denotes s restricted to subinterval $[x_{i-1}, x_i]$ and P_3 is the set of cubic polynomials. The dimension of the Linear Space $S_3([a,b])$ is $N+3$. The set of cubic B-splines $\{B_{-1}(x), B_0(x), \cdots, B_{N+1}(x)\}$ forms a basis for the linear space $S_3([a,b])$ (see [14]), where the cubic B-spline functions $B_i(x)$ at the points $x_0, x_1, \cdots, x_N$ along with the additional points x_{-1} and x_{N+1} can be defined as

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{l-2})^3, & x \in [x_{l-2}, x_{l-1}), \\ (x - x_{l-2})^3 - 4(x - x_{l-1})^3, & x \in [x_{l-1}, x_l), \\ (x_{l+2} - x)^3 - 4(x_{l+1} - x)^3, & x \in [x_l, x_{l+1}), \\ (x_{l+2} - x)^3, & x \in [x_{l+1}, x_{l+2}), \\ 0, & \text{otherwise}. \end{cases}$$