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Weak and Strong Convergence of Two Algorithms for the Split Fixed Point Problem

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Dedicated to Professor Xiaoqing Jin on the occasion of his 60th birthday

Abstract. Two iterative algorithms are proposed for the split fixed point problem. The first algorithm is shown to be weakly convergent and the second one to be strongly convergent. One feature of these algorithms is that the stepsizes are chosen in such a way that no priori knowledge of the operator norms is required. A new idea is introduced in order to prove strong convergence of the second algorithm.

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1. Introduction

We are concerned with algorithmic approaches to the split fixed point problem (SFPP) which is an inverse problem of finding an element in the set of fixed points of one mapping such that its image under a bounded linear operator lies in the set of fixed points of another mapping. More specifically, the SFPP is to

$$\text{find } x \in \mathrm{Fix}(U), \qquad \text{such that } \ Ax \in \mathrm{Fix}(T), \tag{1.1}$$

where H_1 and H_2 are two Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear mappings. Here $\mathrm{Fix}(U)$ and $\mathrm{Fix}(T)$ stand for the fixed point sets of U and T, respectively. In particular, if U and T are orthogonal projections, the SFPP (1.1) is reduced to the split feasibility problem (SFP) [6], which consists of finding a point x with the property:

$$x \in C \quad \text{and} \quad Ax \in Q, \tag{1.2}$$

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where $C \subseteq H_1$ and $Q \subseteq H_2$ are two nonempty closed convex subsets. These problems have been extensively studied in recent years due to their important role in many applied disciplines including signal processing and image reconstruction with particular developments in the medical treatment of intensity-modulated radiation therapy [3,4,8,14,25].

A successful method for solving the SFP (1.2) is Byrne's CQ algorithm [3,4] that generates a sequence $\{x_n\}$ via the iteration process:

$$x_{n+1} = P_C\left[x_n - \tau_n A^*(I - P_Q)Ax_n\right], \tag{1.3}$$

where A^* is the adjoint operator of A, I is the identity operator, τ_n is a real parameter, and P_C and P_Q are the orthogonal projections onto C and Q, respectively. If τ_n is chosen in a compact subset of the interval $\left(0, \frac{2}{\|A\|^2}\right)$, then the CQ algorithm (1.5) converges weakly to a solution of the SFP (1.2) whenever such a solution exists. However, to implement the CQ algorithm, one must compute (or, at least, estimate) the operator norm of A, which is not an easy work in practice, in general. An alternative way of avoiding doing this is to construct variable stepsizes that have no connections with operator norms. Several novel ways of selecting stepsizes have been conducted to overcome this difficulty (see e.g. [14,26]). Among these stepsizes, Yang [26] suggested the following one

$$\tau_n := \frac{\rho_n}{\|A^*(I - P_Q)Ax_n\|}, \tag{1.4}$$

where $\{\rho_n\}$ is a sequence of positive real numbers such that

$$\sum_{k=0}^{\infty} \rho_k = \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \rho_k^2 < \infty. \tag{C1}$$

The CQ algorithm with stepsize (1.4) was first considered in finite dimensional spaces. It is shown in [26] that the CQ algorithm with stepsize (1.4) converges to a solution of the SFP provided that (i) Q is a bounded subset; and (ii) A is a matrix with full column rank. With this choice of the stepsizes, the computation of $\|A\|$ is clearly avoided and hence one need not know *a prior* any information of $\|A\|$. In a recent work, Wang [20] extended this result to Hilbert spaces and completely removed the boundedness condition on Q and the full-column rankness assumption on A.

For solving the SFPP (1.1), Censor and Segal [7] proposed the following method:

$$x_{n+1} = U\left[x_n - \tau_n A^*(I - T)Ax_n\right]. \tag{1.5}$$

This is clearly an extension of the CQ algorithm. The method (1.5) was first considered for the case of quasi firmly nonexpansive operators in finite dimensional spaces. If the parameter τ_n is chosen in a compact subset of the interval $\left(0, \frac{2}{\|A\|^2}\right)$, then the method (1.5) converges weakly to a solution of the SFPP (1.1) whenever such a solution exists. This result was then extended to the case of quasi-nonexpansive operators [15], the case of demicontractive operators [10,16], and the case of finite many quasi firmly nonexpansive operators [5,21]. In the recent work of [2,12], a modification of (1.5) was presented so