Numer. Math. Theor. Meth. Appl. Vol. **4**, No. 1, pp. 13-37
doi: 10.4208/nmtma.2011.m1007 February 2011

Adaptive Hybridized Interior Penalty Discontinuous Galerkin Methods for H(curl)-Elliptic Problems

C. Carstensen[1,2], R. H. W. Hoppe[3,4*], N. Sharma[3] and T. Warburton[5]

[1] *Department of Mathematics, Humboldt Universität zu Berlin, D-10099 Berlin, Germany.*

[2] *Department of Computer Science Engineering, Yonsei University, Seoul 120-749, Korea.*

[3] *Department of Mathematics, University of Houston, Houston TX 77204-3008, USA.*

[4] *Institute of Mathematics, University of Augsburg, D-86159 Augsburg, Germany.*

[5] *CAAM, Rice University, Houston, TX 77005-1892, USA.*

Received 21 April 2010; Accepted (in revised version) 5 October 2010

Available online 29 October 2010

Abstract. We develop and analyze an adaptive hybridized Interior Penalty Discontinuous Galerkin (IPDG-H) method for H(curl)-elliptic boundary value problems in 2D or 3D arising from a semi-discretization of the eddy currents equations. The method can be derived from a mixed formulation of the given boundary value problem and involves a Lagrange multiplier that is an approximation of the tangential traces of the primal variable on the interfaces of the underlying triangulation of the computational domain. It is shown that the IPDG-H technique can be equivalently formulated and thus implemented as a mortar method. The mesh adaptation is based on a residual-type a posteriori error estimator consisting of element and face residuals. Within a unified framework for adaptive finite element methods, we prove the reliability of the estimator up to a consistency error. The performance of the adaptive symmetric IPDG-H method is documented by numerical results for representative test examples in 2D.

AMS subject classifications: 65N30, 65N50, 78M10

Key words: Adaptive hybridized Interior Penalty Discontinuous Galerkin method, a posteriori error analysis, H(curl)-elliptic boundary value problems, semi-discrete eddy currents equations.

1. Introduction

Discontinuous Galerkin (DG) methods are widely used algorithmic schemes for the numerical solution of partial differential equations (PDE). For a comprehensive description, we refer to the survey article [24] and the references therein. As far as elliptic boundary value problems are concerned, DG methods can be derived from a primal-dual mixed formulation using local approximations of the primal and dual variables by polynomial

*Corresponding author. *Email addresses:* cc@math.hu-berlin.de (C. Carstensen), rohop@math.uh.edu (R. H. W. Hoppe), nsharma@math.uh.edu (N. Sharma), timwar@riceu.edu (T. Warburton)

scalar and vector-valued functions and appropriately designed numerical fluxes. Among the most popular schemes are Interior Penalty DG (IPDG) and Local DG (LDG) methods which have been analyzed by means of a priori estimates of the global discretization, e.g., in [3, 5, 23, 39]. For H(curl)-elliptic boundary value problems arising from a semi-discretization of the eddy currents equations, symmetric IPDG methods have been studied in [36]. The time-harmonic Maxwell equations have been addressed in [46].

On the other hand, the a posteriori error analysis and application of adaptive finite element methods (FEM) for the efficient numerical solution of boundary and initial-boundary value problems for PDE has reached some state of maturity as documented by a series of monographs. There exist several concepts including residual and hierarchical type estimators, error estimators that are based on local averaging, the so-called goal oriented dual weighted approach, and functional type error majorants (cf. [2,6,7,30,44,49] and the references therein). A posteriori error estimators for DG methods applied to second order elliptic boundary value problems have been developed and analyzed in [1,11,18,38,40,47]. In particular, a convergence analysis of adaptive symmetric IPDG methods has been provided in [12, 34] and [41]. Residual- and hierarchical-type a posteriori error estimator for H(curl)-elliptic problems have been studied in [8–10, 20, 37]. A convergence analysis for residual estimators has been developed in [19] for 2D and in [35] for 3D problems.

From a computational point of view, DG methods suffer from a relatively huge amount of globally coupled degrees of freedom (DOF) compared to standard FEM. Hybridization is a technique that gives rise to a significant reduction of the globally coupled DOF. It has been introduced for mixed FEM in [31] and further studied in [4,13,15,25,26]. Adaptive mixed hybrid methods on the basis of reliable a posteriori error estimators have been considered in [14, 45] and [50]. For DG methods, a survey of hybridized DG (DG-H) methods has been provided in [26], whereas a unified analysis has been developed in [28]. However, adaptive DG-H methods have not yet been investigated.

In this paper, we will derive and analyze a residual-type a posteriori error estimator for hybridized symmetric IPDG (IPDG-H) methods applied to H(curl)-elliptic boundary value problems in 3D. The analysis will be carried out within a unified framework provided for adaptive finite element approximations in [17, 18, 20–22]. The paper is organized as follows: In Section 2, we introduce some basic notation and present the class of H(curl)-elliptic boundary value problems to be approximated by symmetric IPDG-H methods. Section 3 deals with the development of symmetric IPDG-H methods based on a mixed formulation of the elliptic boundary value problems. We establish its relationship with mortar techniques which allows the implementation as a mortar method. In section 4, we present the residual-type a posteriori error estimator and prove its reliability. Finally, in section 5, we provide a detailed documentation of numerical results to illustrate the performance of the symmetric IPDG-H methods.

2. Basic notations

Let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedral domain with boundary $\Gamma = \partial\Omega$ such that $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \Gamma_D \cap \Gamma_N = \emptyset$. We denote by $\mathscr{D}(\Omega)$ the space of all infinitely often differentiable