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REVIEW ARTICLE

Advances in Studies and Applications of Centroidal Voronoi Tessellations

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Abstract. Centroidal Voronoi tessellations (CVTs) have become a useful tool in many applications ranging from geometric modeling, image and data analysis, and numerical partial differential equations, to problems in physics, astrophysics, chemistry, and biology. In this paper, we briefly review the CVT concept and a few of its generalizations and well-known properties. We then present an overview of recent advances in both mathematical and computational studies and in practical applications of CVTs. Whenever possible, we point out some outstanding issues that still need investigating.

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1. Introduction

A comprehensive study of *centroidal Voronoi tessellations* (CVTs) was provided in the 1999 review article [31]. While the CVT concept initially was phrased as a model and method for optimal point distributions and spacial tessellations of regions/volumes in $\mathbb{R}^d$ or within sets of discrete data, the generality and universality of CVTs have also made them widely applicable in many fields of science and engineering. In the past decade, CVTs and CVT-based methodologies attracted much attentions in the community. Not only significant progress has been made in the theoretical study of the CVTs, but there also

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has been tremendous growth in the scientific and technological applications of CVTs. The purpose of this paper is to reflect upon past progress and to point out some interesting issues that still need to be resolved.

Given the vast literature published on the subject, it is impossible to give a complete survey; instead, we focus on works with which we are most familiar to offer a brief and limited overview of the recent advances in mathematical and computational studies and in practical applications of CVTs. Needless to say, the subject of CVTs is still growing rapidly, so we also point out some outstanding issues and future research topics that remain to be studied.

The paper is organized as follows, in the remainder of this section, we go over the CVT concept and a few of its basic properties. Then, in Section 2, we discuss several generalizations of the basic CVT concept and, in Section 3, we discuss recent progress in the development of improved, i.e., more efficient algorithms, for constructing CVTs. In Section 4, we discuss a very few of the many and ever growing applications to which CVTs have been put to effective use. Brief concluding remarks are given in Section 5.

1.1. The CVT concept in $\mathbb{R}^d$

We first recall the definition of CVTs in Euclidean space. We begin with a given open bounded domain $\Omega \in \mathbb{R}^d$ and a set of distinct points $\{\mathbf{x}_i\}_{i=1}^n \subset \Omega$. For each point $\mathbf{x}_i$, $i = 1, \cdots, n$, define the corresponding *Voronoi region* V_i, $i = 1, \cdots, n$, by

$$V_i = \Big\{ \mathbf{x} \in \Omega \ \mid \ \|\mathbf{x} - \mathbf{x}_i\| < \|\mathbf{x} - \mathbf{x}_j\| \quad \text{for } j = 1, \cdots, n \quad \text{and} \quad j \neq i \Big\}, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean distance (the L^2 metric) in $\mathbb{R}^d$. Clearly $V_i \cap V_j = \emptyset$ for $i \neq j$, and $\cup_{i=1}^n \overline{V}_i = \overline{\Omega}$ so that $\{V_i\}_{i=1}^n$ is a *tessellation* of Ω. We refer to $\{V_i\}_{i=1}^n$ as the *Voronoi tessellation* (VT) of Ω [97] associated with the point set $\{\mathbf{x}_i\}_{i=1}^n$. A point $\mathbf{x}_i$ is called a *generator*; a subdomain $V_i \subset \Omega$ is referred to as the *Voronoi region* (or *Voronoi diagram*) corresponding to the generator $\mathbf{x}_i$. It is well-known that the dual tessellation (in a graph-theoretical sense) to a Voronoi tessellation of Ω is the *Delaunay triangulation* (DT).

Given a density function $\rho(\mathbf{x}) \geq 0$ defined on Ω, for any region $V \subset \Omega$, the standard *mass center* (or *centroid*) $\mathbf{x}^*$ of V is given by

$$\mathbf{x}^* = \frac{\displaystyle\int_V \mathbf{x}\rho(\mathbf{x})\, d\mathbf{x}}{\displaystyle\int_V \rho(\mathbf{x})\, d\mathbf{x}}. \tag{1.2}$$

Then a special family of Voronoi tessellations are defined as follows.

Definition 1. [31] *We refer to a Voronoi tessellation* $\{(\mathbf{x}_i, V_i)\}_{i=1}^n$ *of* Ω *as a* centroidal Voronoi tessellation (CVT) *if and only if the points* $\{\mathbf{x}_i\}_{i=1}^n$ *which serve as the generators of the associated Voronoi regions* $\{V_i\}_{i=1}^n$ *are also the centroids of those regions, i.e., if and only if we have that* $\mathbf{x}_i = \mathbf{x}_i^*$ *for* $i = 1, \cdots, n$. *The corresponding dual triangulation is called a* centroidal Voronoi Delaunay triangulation (CVDT).