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2D Centroidal Voronoi Tessellations with Constraints

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Abstract. We tackle the problem of constructing 2D centroidal Voronoi tessellations with constraints through an efficient and robust construction of bounded Voronoi diagrams, the pseudo-dual of the constrained Delaunay triangulation. We exploit the fact that the cells of the bounded Voronoi diagram can be obtained by clipping the ordinary ones against the constrained Delaunay edges. The clipping itself is efficiently computed by identifying for each constrained edge the (connected) set of triangles whose dual Voronoi vertices are hidden by the constraint. The resulting construction is amenable to Lloyd relaxation so as to obtain a centroidal tessellation with constraints.

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1. Introduction

Voronoi diagrams have been extensively studied in the field of computational geometry [2]. Given a set of points $\mathscr{X} = \{\mathbf{x}_i\}_{i=1}^N$, called sites or generators, the Voronoi diagram is defined as the space decomposition into cells according to the nearest site. Namely, the Voronoi cell associated to $\mathbf{x}_i$, denoted $\mathscr{V}_i$, is defined as

$$\mathscr{V}_i = \left\{ x \in \mathbb{R}^2 \mid d(x, \mathbf{x}_i) \leq d(x, \mathbf{x}_j), \forall j \in \{1, \cdots, N\}, j \neq i \right\}.$$

One popular way to efficiently construct Voronoi diagrams consists in exploiting its duality property with the Delaunay triangulation: The Delaunay triangulation can be defined as the dual of the Voronoi diagram, as the triangulation obtained by creating a Delaunay edge $x_i x_j$ if the Voronoi cells $\mathscr{V}_i$ and $\mathscr{V}_j$ are neighbors (they share a Voronoi edge). A direct characterization of the Delaunay triangulation is also possible: a triangle defined by three points of $\mathscr{X}$ belongs to the Delaunay triangulation if none of the other points of $\mathscr{X}$ is located inside its circumcircle.

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Centroidal Voronoi diagrams are commonly used in some applications which require a good sampling of an input domain. One way to distribute a set of points isotropically and in accordance with a density function is to apply the Lloyd iteration (described in Section 3) over an initial Voronoi diagram of the input points. Du et al. [6] have shown how the Lloyd iteration transforms an initial ordinary Voronoi diagram into a centroidal Voronoi diagram, where each generator happens to coincide with the centroid of its Voronoi cell. This process is a way to trade global requirements (the density function) for local requirements of generating a locally uniform distribution of the Voronoi sites.

The Lloyd iteration assumes a Voronoi tessellation with bounded cells so that the centroid of each Voronoi cell is well defined. Assuming a bounded input domain Ω, one direct way to proceed consists of intersecting each Voronoi cell with Ω and computing the centroid of the resulting intersection (more specifically the connected component containing the cell generator). In addition to suffering from the usual robustness issues, the intersections may result in non-convex or non-simply connected cells and hence in centroids located outside Ω (see Fig. 2-Left & Middle).

For cases where the input domain boundary is a polygonal line, one solution consists of relying on a *constrained Delaunay triangulation* (CDT). It is a generalization of the Delaunay triangulation which allows the addition of constrained line segments appearing as edges of the triangulation [4]. The end points of those line segments are also the generators of the dual Voronoi tessellations. In a CDT, a triangle is valid if its circumcircle does not contain any point of $\mathscr{X}$ visible from inside the triangle. To define the visibility notion, the input domain boundary is considered as a set of occluding barriers (see Fig. 1-Left).

One of our goals is to consider these geometric constraints in a generic manner so as to handle inner isolated constraints, evolving cracks, etc. Among others, natural elements and natural neighbor methods [11, 14, 16, 17] need to handle this type of constraints. In these applications the constraints can move in an unpredictable manner. Robustness is thus a key point of our work, since at every step convex Voronoi cells are required.

We now have to deal with *Voronoi cells with constraints*. As explained above, Delaunay triangulation and Voronoi diagram are dual structures. For our purpose, defining a dual of the CDT would thus be useful. It is possible to construct the usual Voronoi diagram from the Delaunay triangulation with the following dual rule:

> *Voronoi vertices are constructed at circumcenters of Delaunay triangles and Voronoi edges are drawn between dual of neighboring Delaunay triangles.*

The standard dual of the CDT is the *constrained Voronoi diagram* (CVD), defined by a slight modification of this rule:

> *Constrained Voronoi vertices are constructed at circumcenter of constrained Delaunay triangles and constrained Voronoi edges are drawn between dual of Delaunay triangles which are neighbors through a non constrained Delaunay edge*

(see Fig. 1-Middle). Note that this definition allows the Voronoi diagram to cross the constraints, or more exactly, some part of the Voronoi diagram (dashed in Fig. 1-Middle) continue on the wrong side of the constraints as if they were on the right side. Several