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Uniform Convergence Analysis of Finite Difference Scheme for Singularly Perturbed Delay Differential Equation on an Adaptively Generated Grid

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Abstract. Adaptive grid methods are established as valuable computational technique in approximating effectively the solutions of problems with boundary or interior layers. In this paper, we present the analysis of an upwind scheme for singularly perturbed differential-difference equation on a grid which is formed by equidistributing arc-length monitor function. It is shown that the discrete solution obtained converges uniformly with respect to the perturbation parameter. Numerical experiments illustrate in practice the result of convergence proved theoretically.

AMS subject classifications: 65L10, 65L12

Key words: Singular perturbation problems, delay differential equations, boundary layer, upwind scheme, adaptive mesh, uniform convergence.

1. Introduction

In this article, we consider the following singularly perturbed delay differential equation:

$$\begin{cases} \mathscr{L}_\varepsilon u_\varepsilon(x) \equiv -\varepsilon u_\varepsilon''(x) - a(x)u_\varepsilon'(x-\delta) + b(x)u_\varepsilon(x) = f(x), \quad x \in \Omega = (0,1), \\ u_\varepsilon(x) = \gamma(x), \quad -\delta \le x \le 0, \\ u_\varepsilon(1) = \lambda, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is a small parameter and the delay parameter δ is such that $0 < \delta < 1$, which is of $o(\varepsilon)$. The functions $a(x), b(x), f(x)$ and $\gamma(x)$ are sufficiently smooth functions and λ is a constant. It is also assumed that $b(x) \ge \beta > 0, \forall x \in \overline{\Omega}$. Such a problem is sometimes addressed as two-parameter problem. The argument for small delay problems

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are widespread in many mathematical models of biophysics and mechanics where delay term plays an important role in modelling real-life phenomena [12].

When $\delta = 0$, the above equation (1.1) reduces to a singularly perturbed differential equation with a single parameter ε. Depending upon the sign of $a(x)$, *i.e.,* if $a(x) > 0$ (or $a(x) < 0$), a boundary layer is located at left (or right) end of the domain. The layer is maintained for sufficiently small δ with $\delta \neq 0$ and $\delta = o(\varepsilon)$. Lange et al. [5,6] provided an asymptotic approach to boundary value problems (BVP) of the type (1.1). By considering several examples, they have shown that the effect of the small delay on the solution cannot be neglected.

The solution of (1.1) has steep layers which are difficult to approximate efficiently by most numerical methods using uniform grid [4]. In this context, one may think of solving the above problem with a suitably chosen non-uniform grid. If the presence, location, and thickness of a boundary layer is known a priori, then highly appropriate non-uniform grids can be generated. The main disadvantage of this kind of approach is that it relies heavily on knowing a considerable amount about the exact solution before one attempts to solve the differential equation.

A more widely applicable idea is to use an adaptive non-uniform grid where adaptivity is governed by the numerical solution. This approach has the advantage that it can be applied using little or no a priori information. The objective of this paper is to show adaptivity may be used for differential-difference equations (DDE) to generate mesh for which ε-uniform convergence is achieved. With solution-adaptive methods, a commonly used technique for determining the grid points is that they equidistribute a positive monitor function of the numerical solution over the domain. For singular perturbation problems the aim is to cluster automatically grid points within a boundary layer and an obvious choice of adaptivity criterion is therefore the solution gradient [8, 10]. Many authors [2, 11] consider upwind scheme applied to the homogeneous version of (1.1) with $\delta = 0$ (one parameter problems) and $b(x) = 0$ on a non-uniform grid formed by equidistribution of the arc-length monitor function $\sqrt{1 + |u'(x)|^2}$. Their analysis and numerical experiments show that the resulting approximation is indeed first-order uniformly convergent.

A description of the contents of the paper is as follows. In Section 2, we establish the maximum principle for the differential operator, stability result and some a priori estimates on the solution and its derivatives. Section 3 presents upwind finite difference discretization and generation of the non-uniform grids through equidistribution principle. We obtain a bound for the local truncation error in Section 4 and carry out the stability and the error analysis which leads to the main theoretical result namely the ε-uniform convergence in the maximum norm. Finally, several numerical examples are provided in Section 5 to illustrate the applicability of the present method with maximum point-wise error and the rate of convergence is shown in terms of tables and figures. This paper ends with Section 6 that summarizes the main conclusions.

Through out this paper C will denote the generic positive constant independent of the perturbation parameter ε and N (the dimension of the discrete problem), the mesh points x_i, which can take different values at different places, even in the same argument. Here $\|\cdot\|$ denotes the supremum norm over $\overline{\Omega}$.