Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2009.m9001s

Vol. **2**, No. 4, pp. 353-376
November 2009

A Compound Algorithm of Denoising Using Second-Order and Fourth-Order Partial Differential Equations

Qianshun Chang[1], Xuecheng Tai[2,3] and Lily Xing[1,*]

[1] *Institute of Applied Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing, China.*
[2] *School of Physical and Mathematical Sciences, Nanyang Technological University, SPMS-04-01, 21 Nanyang Link, Singapore 637371.*
[3] *Department of Mathematics, University of Bergen, Johannes Brunsgate 12, 5008 Bergen, Norway.*

Received 8 February 2009; Accepted (in revised version) 6 June 2009

Abstract. In this paper, we propose a compound algorithm for the image restoration. The algorithm is a convex combination of the ROF model and the LLT model with a parameter function θ. The numerical experiments demonstrate that our compound algorithm is efficient and preserves the main advantages of the two models. In particular, the errors of the compound algorithm in L_2 norm between the exact images and corresponding restored images are the smallest among the three models. For images with strong noises, the restored images of the compound algorithm are the best in the corresponding restored images. The proposed algorithm combines the fixed point method, an improved AMG method and the Krylov acceleration. It is found that the combination of these methods is efficient and robust in the image restoration.

AMS subject classifications: 68U10, 65M55

Key words: Algorithm of denoising, image restoration, total variation, second-order functional.

1. Introduction

Image restoration is an important problem with numerous applications in both image processing and medical problems. The image restoration is to recover original images from noisy and blurred data. Mathematically, image restoration is to recover the true image u from the observed image z by the formula

$$z = Ku + n, \tag{1.1}$$

where K is a known linear blurring operator and n is a Gaussian white noise.

*Corresponding author. *Email addresses:* `qschang@amss.ac.cn` (Q. Chang), `tai@mi.uib.no` (X. Tai), `xinglily@amss.ac.cn` (L. Xing)

In recent years, many models for noise removal and deblurring are proposed and studied. One of the basic models is the total variation based restoration method proposed by Rudin, Osher and Fatemi [18]. In this model, the following total variation minimization problem is considered:

$$\min_u \left(\alpha \int_\Omega |\nabla u| \, dxdy + \frac{1}{2} \|Ku - z\|_{L^2}^2 \right), \tag{1.2}$$

where $|\nabla u| = \sqrt{u_x^2 + u_y^2}$ and $\alpha > 0$ is the penalty parameter.

The corresponding Euler-Lagrange equation for (1.2) is

$$-\alpha \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + K^*(Ku - z) = 0, \quad \text{in } \Omega, \tag{1.3}$$

where K^* is the adjoint operator of K with respect to standard L_2 inner product. This idea gives a rigorous mathematical tool to introduce nonlinear diffusion filters in the image restoration. Motivated by the total variation norm, many similar models of restoration are proposed in literatures (see [2,3,6,12,13,15,22,23]).

In these papers, several papers (see [6,12,13,15,22,23]) discuss the noise removal methods by fourth-order partial differential equations. Especially, Lysaker, Lundervold and Tai in [13] proposed a problem of second-order functional minimization by the formula

$$\min_u \left(\alpha \int_\Omega |D^2 u| \, dxdy + \frac{1}{2} \|Ku - z\|_{L^2}^2 \right), \tag{1.4}$$

where

$$|D^2 u| = \sqrt{u_{xx}^2 + u_{xy}^2 + u_{yx}^2 + u_{yy}^2}.$$

The corresponding Euler-Lagrange equation for (1.4) is

$$\alpha \left\{ \left(\frac{u_{xx}}{|D^2 u|} \right)_{xx} + \left(\frac{u_{xy}}{|D^2 u|} \right)_{yx} + \left(\frac{u_{yx}}{|D^2 u|} \right)_{xy} + \left(\frac{u_{yy}}{|D^2 u|} \right)_{yy} \right\} + K^*(Ku - z) = 0, \quad \text{in } \Omega. \tag{1.5}$$

It is known that the higher-order PDEs can recover smoother surfaces. In dealing with higher-order PDEs, a major challenge is to pursue the quality in (1.2) along jumps. However, it seems to be difficult to use one algorithm to preserve discontinuities in one part of the image and simultaneously recover smooth signals in other parts. Hence, combining different algorithms remains a possible approach to improve the image restoration capability. In [6,12,15], authors discuss the methods of combining both a lower- and a higher-order PDE.

The ROF model (1.2) is known to be better than LLT model (1.4) when denoising and identifying locations of discontinuities and amplitude of jumps. In another side, the LLT model is better than ROF model in handling smooth signals and keeping small constructions and shapes of the images. In [15], Lysaker and Tai use the convex combination of the solutions of (1.2) and (1.4) to get better restoration images (see [15]).