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# Design of Finite Element Tools for Coupled Surface and Volume Meshes

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**Abstract.** Many problems with underlying variational structure involve a coupling of volume with surface effects. A straight-forward approach in a finite element discretization is to make use of the surface triangulation that is naturally induced by the volume triangulation. In an adaptive method one wants to facilitate "matching" local mesh modifications, i.e., local refinement and/or coarsening, of volume and surface mesh with standard tools such that the surface grid is always induced by the volume grid. We describe the concepts behind this approach for bisectional refinement and describe new tools incorporated in the finite element toolbox ALBERTA. We also present several important applications of the mesh coupling.

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## 1. Introduction

A great variety of problems in science and engineering are modeled mathematically by means of a system of partial differential equations (PDEs) closed with suitable initial, boundary, or interface conditions. The PDEs are defined on a domain in space or space-time and in many applications the shape of the domain may also be unknown beforehand, and must be determined as part of the solution. In addition, the problems under consideration involve a coupling of surface and bulk effects. The mathematical description may reflect this in that the PDEs contain some unknowns defined on a spacial domain $\Omega$ as well as other unknowns defined on a lower-dimensional manifold $\Gamma \subset \bar{\Omega}$, for instance the domain boundary $\partial\Omega$. In Section 2 we give several examples of such problems.

These problems may be numerically solved using various discretization schemes and techniques. In this paper we will focus on finite element discretizations. Most finite element methods for time-dependent problems do not mesh the space-time domain, but

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employ a suitable time discretization for converting the time-dependent problem into a sequence of stationary problems. This allows us to restrict ourselves to spacial domains. Furthermore, we only consider simplicial grids but the derived methods directly carry over to other types of meshes.

During the last decades, adaptive finite elements have become a well-established tool for the numerical solution of boundary value problems, see the monographs [1, 5, 59] and the references therein. Adaptivity is designed to use computational resources more efficiently. In higher space dimensions some problems may only be solvable in reasonable time using adaptive methods. Adaptive finite element methods employ an iteration of the form

$$\mathsf{SOLVE} \quad \to \quad \mathsf{ESTIMATE} \quad \to \quad \mathsf{MARK} \quad \to \quad \mathsf{REFINE/COARSEN}$$

for adapting the finite element mesh to the solution of the underlying problem. Given a grid, procedure SOLVE computes the discrete solution and ESTIMATE computes an a posteriori error estimate, which is an upper bound for the error in some given norm in terms of the discrete solution and data of the PDE. Usually, the estimator is built from element error indicators, which are used in MARK for selecting elements subject to refinement and/or coarsening. In the last step, refinement and/or coarsening algorithms locally refine and/or coarsen the grid based on the decisions taken in MARK, see for instance [49] for a more detailed description. For elliptic problems the above adaptive loop is well analyzed with respect to convergence [20, 41, 42] and optimal cardinality [9, 14, 53].

The finite element discretization of problems involving bulk and surface effects is done by triangulating $\Omega$ as well as $\Gamma$ and defining finite element spaces on both triangulations. The different spaces are then used for approximating bulk quantities respectively surface quantities. The surface triangulation is naturally defined by collecting the faces of elements of the bulk triangulation that lie on $\Gamma$, i.e., the surface grid is the trace of the volume grid. Since bulk and surface effects interact, we need restrictions of bulk quantities to the surface, naturally introducing the concept of trace spaces. Some applications may also require prolongations of surface quantities to the bulk. For standard Lagrange finite element discretizations both tasks are facilitated by an injective mapping connecting surface degrees of freedom (DOFs) with bulk degrees of freedom. Such a mapping in combination with corresponding finite element bases for bulk and surface then exactly realizes the finite element space on $\Gamma$ as the trace space of the bulk space defined over $\Omega$.

Coupled meshes are easily handled if the meshes do not change during a computation. In the setting of adaptive methods the problems of coupling grids become inherently more complex. If an adaptive method requires a change of any of the involved meshes, we might lose the useful property that the surface grids were originally defined as the collection of bulk faces. Without this property the transfer of data between bulk and surface meshes becomes much more difficult. In this scenario, after each mesh change one would have to somehow reconstruct the connection of bulk elements with surface elements, a cumbersome, possibly costly process. The aforementioned mapping of DOFs would no longer exist and would have to established from scratch.

It is thus highly desirable that the coupling is maintained automatically during local mesh modifications, i.e., refinement and coarsening of relatively small patches of elements.