Finite-Difference Methods for a Class of Strongly Nonlinear Singular Perturbation Problems

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Abstract. The paper is concerned with strongly nonlinear singularly perturbed boundary value problems in one dimension. The problems are solved numerically by finite-difference schemes on special meshes which are dense in the boundary layers. The Bakhvalov mesh and a special piecewise equidistant mesh are analyzed. For the central scheme, error estimates are derived in a discrete $L^1$ norm. They are of second order and decrease together with the perturbation parameter $\epsilon$. The fourth-order Numerov scheme and the Shishkin mesh are also tested numerically. Numerical results show $\epsilon$-uniform pointwise convergence on the Bakhvalov and Shishkin meshes.

AMS subject classifications: 65L10, 65L12

Key words: Boundary-value problem, singular perturbation, finite differences, Bakhvalov and piecewise equidistant meshes, $L^1$ stability.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

We consider the following singularly perturbed boundary value problem:

$$-\epsilon^2 (k(x)u')' + c(x,u) = 0, \quad x \in I := [0,1], \quad u(0) = \alpha, \quad u(1) = \beta,$$  (1.1)

where $\epsilon$ is a small positive parameter, $\alpha$ and $\beta$ are given constants, and the functions $k$ and $c$ are sufficiently smooth and satisfy

$$k^* \geq k(x) \geq k_* > 0, \quad c_u(x,u) \geq c_* > 0, \quad x \in I, \quad u \in \mathbb{R}.$$  (1.2)

This problem has a unique solution, $u_\epsilon$, for which the following estimates hold true:

$$|u_\epsilon^{(j)}(x)| \leq M \left( 1 + \epsilon^{-j} e^{-\gamma x/\epsilon} + \epsilon^{-j} e^{\gamma(x-1)/\epsilon} \right), \quad x \in I, \quad j = 0,1,2,3,4,$$  (1.3)

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with a constant $\gamma$ in the interval $(0, \sqrt{c_s/k^2})$. Here and throughout the paper, $M$ is a
generic positive constant independent of $\varepsilon$. Thus, estimates (1.3) show that the solution
has in general two boundary layers whose width is $O(\varepsilon \ln \frac{1}{\varepsilon})$. This result can be proved
as follows. For $K(u) = \int_0^u k(s) \, ds$, it holds that $K(u) \geq k_0 > 0$, so the inverse function $K^{-1}$
exists. We can therefore introduce the substitution $\nu = K(u)$ to transform (1.1) to
\begin{equation}
-\varepsilon^2 \nu'' + g(x, \nu) = 0, \quad x \in I, \quad \nu(0) = K(\alpha), \quad \nu(1) = K(\beta),
\end{equation}
where $g(x, \nu) = c(x, K^{-1}(\nu))$. Then from $g_x(x, \nu) = c_u(x, K^{-1}(\nu))/k(u)$, we get that
$g_x(x, \nu) > \gamma^2$. This implies that problem (1.4) has a unique solution, $\nu_\varepsilon$, and it is well
known that its derivatives can be estimated by the right-hand side of (1.3). These estimates
immediately transfer to $u_\varepsilon$.

Problems similar to (1.1), as well as the more general ones with $k = k(x, u)$, arise
in applications to chemistry as models of catalytic reactions accompanied by a change in
volume $[3, 14, 17, 19]$. Some numerical methods for those problems have been considered
in $[14, 17]$, but no complete error-analysis has been given. This is finally done in the
present paper. The special case $k(u) \equiv 1$ describes the standard reaction-diffusion problem
which has been discussed very often. Earlier papers, like $[2, 13]$, typically consider the
condition $c_u(x, u) \geq c_s > 0$, which is also assumed here. This condition is relaxed in
$[7, 8, 12, 15]$. Of other more recent papers on numerical methods for singularly perturbed
semilinear reaction-diffusion problems, let us mention $[5]$ and $[6]$. These papers deal with

The numerical method proposed by Wang $[18]$ for (1.1) in the non-perturbed case
$\varepsilon = 1$ is the fourth-order Numerov scheme applied to (1.4). Wang considers the situation
when $K^{-1}$ can be found explicitly. Since this is not always easy to do, we discretize here
the original problem after rewriting the differential equation in (1.1) as
\begin{equation}
-\varepsilon^2 K(u)'' + c(x, u) = 0.
\end{equation}
The method we discuss in detail is the central finite-difference scheme applied on meshes of
Bakhvalov and piecewise equidistant types. It is well known in the semilinear case $k(u) \equiv 1$
that the central scheme is $\varepsilon$-uniformly stable in the maximum norm. Here, because of the
strong nonlinearity of the problem, it is much easier to use a discrete $L^1$ norm to prove
stability uniform in $\varepsilon$. Stability of finite-difference approximations of quasilinear singular
perturbation problems is often proved in this norm, see $[1]$ for instance. Solutions of such
problems may have interior layers with a priori unknown locations. This is not the case in the
present problem, but, in addition to the strong nonlinearity, there is another reason for
using the $L^1$ norm. If $w(x) = \exp(-\gamma x/\varepsilon)$ is the exponential boundary-layer function, then
$\|w\|_1$ is of order $\varepsilon$, thus small values of $\varepsilon$ increase accuracy in $L^1$ norm. Such higher $L^1$-
accuracy is important in the catalytic-reaction applications when calculating the so-called
efficiency factor, see $[17]$.

$\varepsilon$-uniform stability in $L^1$ norm implies convergence results in the same norm, the errors
being estimated by
\begin{equation}
E_B := MN^{-2} \left( \varepsilon + \varepsilon^{-mN} \right) \text{ on the Bakhvalov mesh}
\end{equation}