NUMERICAL MATHEMATICS: Theory, Methods and Applications
Numer. Math. Theor. Meth. Appl., Vol. **1**, No. 2, pp. 113-137 (2008)

An Adaptive Uniaxial Perfectly Matched Layer Method for Time-Harmonic Scattering Problems

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Received 1 July, 2007; Accepted (in revised version) 26 November, 2007

Abstract. The uniaxial perfectly matched layer (PML) method uses rectangular domain to define the PML problem and thus provides greater flexibility and efficiency in dealing with problems involving anisotropic scatterers. In this paper an adaptive uniaxial PML technique for solving the time harmonic Helmholtz scattering problem is developed. The PML parameters such as the thickness of the layer and the fictitious medium property are determined through sharp a posteriori error estimates. The adaptive finite element method based on a posteriori error estimate is proposed to solve the PML equation which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer. Numerical experiments are included to illustrate the competitive behavior of the proposed adaptive method. In particular, it is demonstrated that the PML layer can be chosen as close to one wave-length from the scatterer and still yields good accuracy and efficiency in approximating the far fields.

AMS subject classifications: 65N30, 65N50
Key words: Adaptivity, uniaxial perfectly matched layer, a posteriori error analysis, acoustic scattering problems.

Dedicated to Professor Yucheng Su on the Occasion of His 80th Birthday

1. Introduction

We propose and study a uniaxial perfectly matched layer (PML) technique for solving Helmholtz-type scattering problems with perfectly conducting boundary:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \tag{1.1a}$$

$$\frac{\partial u}{\partial \mathbf{n}_D} = -g \quad \text{on } \Gamma_D, \tag{1.1b}$$

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{1.1c}$$

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Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ is determined by the incoming wave, and $\mathbf{n}_D$ is the unit outer normal to Γ_D. We assume the wave number $k \in \mathbb{R}$ is a constant. We remark that the results in this paper can be extended to the case when $k^2(x)$ is a variable wave number inside some bounded domain, or to solve the scattering problems with other boundary conditions, such as Dirichlet or the impedance boundary condition on Γ_D.

Since the work of Bérénger [3] which proposed a PML technique for solving the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf., e.g., Turkel and Yefet [19], Teixeira and Chew [18] for the reviews). The basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain.

The convergence of the PML method is studied in Lassas and Somersalo [13], Hohage et al. [12] for the acoustic scattering problems for circular PML layers and in Lassas and Somersalo [14] for general smooth convex geometry. It is proved in [12–14] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinite. We remark that in practical applications involving PML techniques, one cannot afford to use a very thick PML layer if uniform meshes are used because it requires excessive grid points and hence more computer time and more storage. On the other hand, a thin PML layer requires a rapid variation of the artificial material property which deteriorates the accuracy if too coarse mesh is used in the PML layer.

The adaptive PML technique was proposed in Chen and Wu [8] for a scattering problem by periodic structures (the grating problem), in Chen and Liu [6] for the acoustic scattering problem, and in Chen and Chen [5] for Maxwell scattering problems. The main idea of the adaptive PML technique is to use the a posteriori error estimate to determine the PML parameters and to use the adaptive finite element method to solve the PML equations. The adaptive PML technique provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

The purpose of this paper is to extend the adaptive PML technique developed for circular PML layer in [5, 6, 8] to deal with the uniaxial PML methods which are widely used in the engineering literature. The main advantage of the uniaxial PML method as opposing to the circular PML method is that it provides greater flexibility and efficiency to solve problems involving anisotropic scatterers. Our technique to prove the PML convergence is different from the techniques developed in [5, 6, 8] for circular PML layers. It is based on the integral representation of the exterior Dirichlet problem for the Helmholtz equation and the idea of the complex coordinate stretching. To the authors' best knowledge, this is the first convergence proof of the uniaxial PML method in the literature. We remark that the boundary of the uniaxial PML layer is only Lipschitz and so the results in [14] cannot be applied.