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A High Order Operator Splitting Method for the Degasperis–Procesi Equation

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Abstract. The Degasperis–Procesi equation is split into a system of a hyperbolic equation and an elliptic equation. For the hyperbolic equation, we use the high order finite difference WENO-Z scheme to approximate the nonlinear flux. For the elliptic equation, the wavelet collocation method is employed to discretize the high order derivative. Due to the combination of the WENO-Z reconstruction and the wavelet collocation, the splitting method shows an excellent ability in capturing the formation and propagation of shockpeakon solutions. The numerical simulations for different solutions of the Degasperis–Procesi equation are conducted to illustrate high accuracy and capability of the proposed method.

AMS subject classifications: 65M10, 78A48

Key words: Degasperis–Procesi equation, discontinuous solution, weighted essentially non-oscillatory method, wavelet collocation method.

1. Introduction

In this paper, we consider numerical approximations to the Degasperis–Procesi (DP) equation

$$u_t - u_{xxt} + 4f(u)_x + 3\kappa^3 u_x = f(u)_{xxx}, \tag{1.1}$$

where $f(u) = u^2/2$. This integrable partial differential equation modeling the propagation of nonlinear dispersive waves [1] was first found by Degasperis and Procesi when they were studying the third order dispersive equation [2]

$$u_t + \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx}\right)_x, \tag{1.2}$$

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with six real constants $c_0, c_1, c_2, c_3, \gamma, \alpha \in \mathbb{R}$. By applying the method of asymptotic integrability, they confirmed that only three equations are asymptotically integrable among this family, including the Korteweg–de Vries (KdV) equation ($\alpha = c_2 = c_3 = 0$), the Camassa–Holm (CH) equation $\left(c_1 = -\frac{3c_3}{2\alpha^2}, c_2 = \frac{c_3}{2}\right)$ and the DP equation (1.1). By rescaling, shifting the dependent variable and using a Galilean boost [1], one can observe that the DP equation has a similar form to the limiting case of the CH equation. Actually, both of them can be viewed as the models of shallow water waves by hodograph transformations. The former one is related to the AKNS shallow water wave equation [3], and the latter one is related to the Hirota–Satsuma shallow water wave equation [4].

Despite the similarities to the CH equation, we would like to point out that the DP equation is truly different. One of the major features of the DP equation ($\kappa = 0$) is that it has not only peak solutions [1], $u(x,t) = ce^{-|x-ct|}$, but also a shock wave solution of the form [5, 6]

$$u(x,t) = ce^{-|x-ct|} + \frac{s}{ts+1}\mathrm{sign}(x-ct)e^{-|x-ct|}, \tag{1.3}$$

where $c, s (s > 0)$ are constants. Besides the solution discontinuity occurs possibly in the DP equation but not at all in the CH equation, the invariants embedded in the DP equation are much weaker than those of the CH equation [7]:

- Three useful invariants of the DP equation are

$$I_1(u) = \int_R u\mathrm{d}x, \quad I_2(u) = \int_R u^3\mathrm{d}x, \quad I_3(u) = \int_R (u - u_{xx})v\mathrm{d}x, \tag{1.4}$$

  where $4v - v_{xx} = u$.

- Three useful invariants of the CH equation are

$$H_1(u) = \int_R (u - u_{xx})\mathrm{d}x, \quad H_2(u) = \int_R (u^2 + u_x^2)\mathrm{d}x, \quad H_3(u) = \int_R (u^3 + uu_x^2)\mathrm{d}x. \tag{1.5}$$

The invariant H_2 plays an important role in the analysis and development of numerical schemes for the CH equation. However, the corresponding invariants of the DP equation can not control the H^1 norm [8]. Particularly, the lack of smoothness in the solution introduces more difficulty in the numerical computation. It is a challenge to design stable and high order accurate numerical schemes for solving this equation.

In the last decades, a lot theoretical studies have been carried out for the DP equation. Coclite and Karlsen showed that the DP equation could admit discontinuous (shock wave) solutions and proved the existence and uniqueness results for entropy weak solutions belonging to the class $L^1 \cap BV$ [6]. For the well-posedness of the initial value problem, Yin gave a discussion within certain functional classes in a series of papers [9–11]. And some blow-up results were obtained in [12, 13]. Explicit or special solutions were also studied, for example, explicit multipeakon solutions [14], entropy shock solutions [13], the general N-soliton solutions [15], traveling wave solutions [16] and so on.