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Convergence Analysis of Stochastic Collocation Methods for Maxwell Equations with Random Inputs

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Abstract. In this paper, we consider a stochastic collocation method for the Maxwell equations with random inputs. We first verify the regularity of the solutions for the model problem, which depends on the random dielectric constant, random magnetic permeability and the initial and boundary data. Then the convergence of our numerical approach is proved. Further some numerical examples are presented to support the analysis.

AMS subject classifications: 35Q61, 65C30, 65M70

Key words: Maxwell equations, convergence analysis, stochastic collocation methods, Lagrange interpolation, regularity.

1. Introduction

In simulating complex physical or engineering systems, there usually exist some uncertain factors in the model coefficients, constitutive laws, boundary and initial conditions, forcing terms, and geometric irregularities of the physical problems. In order to acquire reliable numerical predictions, one has to take the uncertainty quantification caused by the random input data into account. Recently some numerical methods have been developed for solving partial differential equations with random inputs, such as Monte Carlo and sampling based method [4, 5], perturbation method [14] and two-level stochastic collocation method [3]. Especially, the generalized polynomial chaos (gPC) method [25] has been broadly used to solve PDEs with random inputs. With the gPC, stochastic solutions are expressed as orthogonal polynomials of the input random parameters, and different types of orthogonal polynomials can be chosen by different random distributions. Actually it is essentially a spectral representation in random space, and exhibits fast convergence when

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the solution depends smoothly on the random parameters. In recent years, more and more attentions have been paid to solve the hyperbolic equations with random inputs. Gottlieb and Xiu [9] made the first attempt to consider a simple hyperbolic model of a scalar wave equation with random wave speed using stochastic Galerkin method. Tang and Zhou [19] analyzed the regularity of the model problem in [9] and verified the convergence of the stochastic collocation method. Hu and Jin [12] developed a stochastic Galerkin method for the Boltzmann equation with uncertainty.

In this paper, a stochastic collocation method will be implemented to solve the Maxwell equations in 2-D case with random inputs. Actually collocation methods have been used in different disciplines for uncertainty quantification (see, e.g., [6, 7, 10, 11, 13, 15, 16, 21, 24, 26]). In collocation approaches one seeks numerical solutions satisfying the governing differential equations at a set of nodes, which are taken as sample points in the corresponding random space. Two of the main high-order stochastic collocation approaches are the Lagrange interpolation approach [24] and the pseudo-spectral gPC approach [23]. Following the strategies introduced by Tatang [21], we take the roots of some orthogonal polynomial as the nodes at which the approximate solution is to be found. Let $\Theta = \{\theta_k\}_{k=0}^{N}$ be such a set of nodes, where N is the number of them. A Lagrange interpolation of the solution $w(\theta)$ can be written as

$$I^N w(\theta) = \sum_{k=0}^{N} \tilde{w}_k F_k(\theta),$$

where $F_k(\theta) \in P_N$ is the Lagrange interpolation basis function satisfying

$$F_k(\theta_i) = \delta_{ki}, \quad 0 \le i, \quad k \le N,$$

and $\tilde{w}_k := w(\theta_k), 0 \le k \le N$, the value of w at $\theta_k \in \Theta$.

We consider the following Maxwell equations with random inputs:

$$\begin{cases} \dfrac{\partial E_x}{\partial t} = \dfrac{1}{\varepsilon^*(\theta)} \dfrac{\partial H_z}{\partial y}, \\ \dfrac{\partial E_y}{\partial t} = -\dfrac{1}{\varepsilon^*(\theta)} \dfrac{\partial H_z}{\partial x}, \\ \dfrac{\partial H_z}{\partial t} = \dfrac{1}{\mu^*(\theta)} \Big(\dfrac{\partial E_x}{\partial y} - \dfrac{\partial E_y}{\partial x} \Big), \end{cases} \tag{1.1}$$

where $\varepsilon^*(\theta)$ and $\mu^*(\theta)$ are the stochastic dielectric coefficient and stochastic magnetic permeability of the medium respectively, $\big(E_x(x, y, t; \theta), E_y(x, y, t; \theta)\big)$ are the functions for the electric field, and $H_z(x, y, t; \theta)$ is the function for the magnetic field, $(x, y) \in D = [0, a] \times [0, b]$ and the random variable $\theta \in \Gamma$ with D the physical space and Γ the random space. The corresponding initial conditions are given by

$$E_x(x, y, 0; \theta) = E_{x_0}, \quad E_y(x, y, 0; \theta) = E_{y_0}, \quad H_z(x, y, 0; \theta) = H_{z_0}. \tag{1.2}$$

Let (Ω, A, P) be a complete probability space with Ω the set of outcomes, A the σ-algebra of events, and P a probability measure. Denote $\rho(\theta) : \Gamma \to R^+$ as the probability density