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Spectral Distribution in the Eigenvalues Sequence of Products of g-Toeplitz Structures

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Abstract. Starting from the definition of an $n \times n$ g-Toeplitz matrix, $T_{n,g}(u) = \left[\widehat{u}_{r-gs}\right]_{r,s=0}^{n-1}$, where g is a given nonnegative parameter, $\{\widehat{u}_k\}$ is the sequence of Fourier coefficients of the Lebesgue integrable function u defined over the domain $\mathbb{T} = (-\pi, \pi]$, we consider the product of g-Toeplitz sequences of matrices $\{T_{n,g}(f_1)T_{n,g}(f_2)\}$, which extends the product of Toeplitz structures $\{T_n(f_1)T_n(f_2)\}$, in the case where the symbols $f_1, f_2 \in L^\infty(\mathbb{T})$. Under suitable assumptions, the spectral distribution in the eigenvalues sequence is completely characterized for the products of g-Toeplitz structures. Specifically, for $g \geq 2$ our result shows that the sequences $\{T_{n,g}(f_1)T_{n,g}(f_2)\}$ are clustered to zero. This extends the well-known result, which concerns the classical case (that is, $g = 1$) of products of Toeplitz matrices. Finally, a large set of numerical examples confirming the theoretic analysis is presented and discussed.

AMS subject classifications: 65F10, 15A18, 47B36, 47B65

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1. Introduction

Let f be a Lebesgue function defined on the interval $(-\pi, \pi]$. We recall that for a given nonnegative integer g, an $n \times n$ matrix $A_{n,g}$ is called g-Toeplitz if $A_{n,g} = \left[\hat{f}_{r-gs}\right]_{r,s=0}^{n-1}$. In this case, a g-Toeplitz matrix is denoted by $T_{n,g}(f)$ and the sequence $\{\hat{f}_k\}_k$ of entries of $T_{n,g}(f)$ can be interpreted as the sequence of Fourier coefficients of an integrable function f defined on $\mathbb{T}$. In this work we are motivated by the variety of fields where such matrices can be encountered, e.g., multigrid methods [14], wavelet analysis together with the subdivision algorithms, or equivalently, in the associated refinement equations, see [9, 10] and the references therein. Furthermore, interesting connections between dilation equations in the wavelets context and multigrid algorithms [14, 43] were proven by Gilbert Strang [39] when establishing the restriction/prolongation operators [1,12] with boundary conditions.

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The use of different boundary conditions is quite natural when treating with signal/image restoration problems or differential equations, see [27,28].

We denote the usual Hilbert space of square-integrable functions over the circle $G = \{z \in \mathbb{C},\ |z| = 1\}$ by $L^2(G)$, and let $\mathscr{H}^2$ be the Hardy space of functions belonging to $L^2(G)$, and whose the negative Fourier coefficients are equal to zero. Obviously, the subset G is isomorphic to the set $\mathbb{T}$, and the notation $G \cong \mathbb{T}$ means that both domains G and $\mathbb{T}$ are isomorph. In the rest of this paper we sometimes use the domain G or $\mathbb{T}$, depending on the context. Let us define the g-Toeplitz operator $T_{f,g}$ with generating function f, as the operator

$$\begin{aligned} T_{f,g} : \mathscr{H}^2 &\to \mathscr{H}^2, \\ u &\mapsto P_g(fu), \end{aligned}$$

where P_g is the mapping from $L^2(G)$ onto $\mathscr{H}^2$ defined as

$$P_g(fu) := P^{\perp}(fu_g), \tag{1.1}$$

where $u_g \in \mathscr{H}^2$ completely depends on the parameter g and the function u. For example, if u is defined on $\mathbb{T}$ by $u(t) = \exp(it)$, then the function u_g is given by

$$u_g(t) = \exp(igt), \quad \forall t \in \widehat{\mathbb{T}} = \left(-\frac{\pi}{g}, \frac{\pi}{g}\right]. \tag{1.2}$$

More specifically, $u_g = u \circ h_g$, where h_g is the map from $\widehat{\mathbb{T}}$ onto $\mathbb{T}$ defined as $h_g(t) = gt$. Furthermore, $P^{\perp}$ is the orthogonal projection from $L^2(G)$ onto $\mathscr{H}^2$. It is worth noticing that such an operator, $T_{f,g}$, is bounded if and only if the symbol f is in the space of (essentially) bounded functions on the circle, and its infinite matrix, $T_g(f)$, in the canonical orthonormal basis $\mathscr{B} = \{1, z, z^2, \cdots\}$ is not (in general) constant along the diagonals, whenever $g > 1$. More specifically, the entries of $T_g(f)$ obey the rule $T_g(f) = \left[\hat{f}_{r-gs}\right]_{r,s=1}^{\infty}$, where the entries $\hat{f}_k$ can be interpreted as the Fourier coefficients of the symbol f defined by

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\exp(-ikt)dt. \tag{1.3}$$

Now, let $u \in L^1(\mathbb{T})$ and let n be a non-negative integer. By $T_{n,g}(u)$ we denote the $n \times n$ matrix $\left[\hat{u}_{r-gs}\right]_{r,s=1}^{n}$. It is not hard to prove that the sequence of operators on $\mathscr{H}^2$, associated with the sequences $\{T_{n,g}(u)\}_{n=1}^{\infty}$, is an approximating sequence for the g-Toeplitz operator $T_{u,g}$, when $u \in L^{\infty}(G)$ (the space of (essentially) bounded functions on the circle), hence $\{T_{n,g}(u)\}_{n=1}^{\infty}$ is called a g-Toeplitz sequence. It is interesting to ask how the spectrum, $\Lambda_{n,g} = \{\lambda_1, \cdots, \lambda_n\}$, of $T_{n,g}(u)$ is associated with the set of the eigenvalues of $T_g(u)$ if $u \in L^{\infty}(G)$, or even if $u \in L^1(G)$, to analyze the "spectral behavior" of the sequence of sets $\{\Lambda_{n,g}\}_{n=1}^{\infty}$ (or that of the sequence $\{\Gamma_{n,g}\}_{n=1}^{\infty}$, where $\Gamma_{n,g}$ represents the set of singular values of $T_{n,g}(u)$). When $g = 1$, $T_{n,1}(u)$ is nothing but the classical Toeplitz matrix $T_n(u)$, so an