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An Interface-Unfitted Finite Element Method for Elliptic Interface Optimal Control Problems

Chaochao Yang[1,2], Tao Wang[1] and Xiaoping Xie[1,*]

[1] *School of Mathematics, Sichuan University, Chengdu 610064, China*
[2] *School of Science, Chongqing University of Technology, Chongqing 400054, China*

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Abstract. This paper develops and analyses numerical approximation for linear-quadratic optimal control problems governed by elliptic interface equations. We adopt variational discretization concept to discretize optimal control problems, and apply an interface-unfitted finite element method due to [A. Hansbo and P. Hansbo. An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. Comput. Methods Appl. Mech. Engrg., 191(47-48): 5537-5552, 2002] to discretize the corresponding state and adjoint equations, where piecewise cut basis functions around interface are enriched into standard conforming finite element space. Optimal error estimates in both L^2 norm and a mesh-dependent norm are derived for the optimal state, co-state and control under different regularity assumptions. Numerical results verify the theoretical results.

AMS subject classifications: 49J20, 49M25, 65N12, 65N30

Key words: Interface equations, interface control, variational discretization concept, cut finite element method.

1. Introduction

Many optimization processes in science and engineering lead to optimal control problems governed by partial differential equations (PDEs). In particular in some practical problems, such as the multi-physics progress or engineering design with different materials, the corresponding controlled systems are described by elliptic equations with interface, whose coefficients are discontinuous across the interface.

Let us consider the following linear-quadratic optimal control problem governed by elliptic interface equations:

$$\min J(y,u) := \frac{1}{2}\int_\Omega (y-y_d)^2\,dx + \frac{\alpha}{2}\int_\Gamma u^2\,ds \tag{1.1}$$

*Corresponding author. *Email addresses:* `yangchaochao9055@163.com` (C. C. Yang), `602721197@qq.com` (T. Wang), `xpxie@scu.edu.cn` (X. P. Xie)

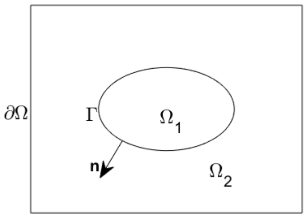

Figure 1: The geometry of an interface problem: an illustration.

for $(y,u)\in H_0^1(\Omega)\times L^2(\Gamma)$ subject to the elliptic interface problem

$$\begin{cases} -\nabla\cdot(a(x)\nabla y)=f & \text{in } \Omega, \\ y=0 & \text{on } \partial\Omega, \\ [y]=0, [a\nabla_n y]=g+u & \text{on } \Gamma \end{cases} \tag{1.2}$$

with the control constraint

$$u_a\le u\le u_b, \qquad \text{a.e. on } \Gamma. \tag{1.3}$$

Here $\Omega\subseteq\mathbb{R}^d(d=2,3)$ is a polygonal or polyhedral domain, consisting of two disjoint subdomains $\Omega_i(1\le i\le 2)$, and interface $\Gamma=\partial\Omega_1\cap\partial\Omega_2$; see Fig. 1 for an illustration. $y_d\in L^2(\Omega)$ is the desired state to be achieved by controlling u through interface Γ, and α is a positive constant. $a(\cdot)$ is piecewise constant with

$$a|_{\Omega_i}=a_i>0, \quad i=1,2.$$

$[y]:=(y|_{\Omega_1})|_\Gamma-(y|_{\Omega_2})|_\Gamma$ is the jump of function y across interface Γ, $\nabla_n y=n\cdot\nabla y$ is the normal derivative of y with n denoting the unit outward normal vector along $\partial\Omega_1\cap\Gamma$,

$$f\in L^2(\Omega), \quad g\in H^{1/2}(\Gamma) \quad \text{and} \quad u_a,u_b\in H^{1/2}(\Gamma) \;\; \text{with } u_a\le u_b \text{ a.e. on } \Gamma. \tag{1.4}$$

The choice of homogeneous boundary condition on boundary $\partial\Omega$ is made for ease of presentation, since similar results are valid for other boundary conditions.

For the elliptic interface problem, the global regularity of its solution is often low due to the discontinuity of coefficient $a(\cdot)$. The low global regularity may result in reduced accuracy for its finite element approximations [1, 55], especially when the interface has complicated geometrical structure [29, 40]. Generally there have two categories in the literature to tackle this difficulty, i.e. interface(or body)-fitted methods [2, 7, 11, 15, 16, 28, 33, 46, 56, 59] and interface-unfitted methods. For the interface-fitted methods, meshes aligned with the interface are used so as to dominate the approximation error caused by the non-smoothness of solution. In practice, it is usually difficult to construct such meshes, especially in three-dimensional problems.

In contrast, the interface-unfitted methods, with certain types of modifications for approximating functions around interface, do not require the meshes to fit the interface, and thus avoid complicated mesh generation. For some representative interface-unfitted methods, we refer to the extended/generalized finite element method [5, 42–44, 51], where