Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2018-0026

Vol. **12**, No. 3, pp. 681-708
August 2019

An Unconditionally Stable Laguerre Based Finite Difference Method for Transient Diffusion and Convection-Diffusion Problems

Wescley T. B. de Sousa[1,*] and Carlos F. T. Matt[2]

[1] *Institute for Technical Physics, Karlsruhe Institute of Technology, 76344 Eggenstein-Leopoldshafen, Germany*
[2] *CEPEL - Electric Power Research Center (ELETROBRAS System), Horácio Macedo Avenue, 354, 21941-911 Rio de Janeiro, Brazil*

Received 21 February 2018; Accepted (in revised version) 15 May 2018

Abstract. This paper describes an application of weighted Laguerre polynomial functions to produce an unconditionally stable and accurate finite-difference scheme for the numerical solution of transient diffusion and convection-diffusion problems. The unconditionally stability of Laguerre-FDM (L-FDM) is guaranteed by expanding the time dependency of the unknown potential as a series of orthogonal functions in the domain $(0,\infty)$, avoiding thus any time integration scheme. The L-FDM is a marching-on-in-degree scheme instead of traditional marching-on-in-time methods. For the two heat-transfer problems, we demonstrated the accuracy, numerical stability and computational efficiency of the proposed L-FDM by comparing its results against closed-form analytical solutions and numerical results obtained from classical finite-difference schemes as, for instance, the Alternating Direction Implicit (ADI).

AMS subject classifications: 34B60, 65N06, 78M20, 33C45

Key words: Finite difference scheme, Laguerre polynomials, numerical methods, diffusion and convection-diffusion problems.

1. Introduction

The conventional finite difference method (FDM) has been proven to be an effective technique for accurate solution of numerous transient diffusion and convection-diffusion problems. Finite-difference (FD) schemes for transient problems may be broadly classified into explicit and implicit schemes, and a combination thereof. Explicit FD schemes are easier to implement as the unknown potential at the future time step may be written explicitly as a function of its values on previous time step. Conversely, implicit FD schemes require at each time step the solution of a coupled system of algebraic equations and thus higher computational effort.

*Corresponding author. *Email addresses:* sousa@kit.edu (W. T. B. de Sousa), cfmatt@cepel.com (C. F. T. Matt)

Explicit FD schemes are conditionally stable. For transient diffusion problems, numerical stability is only guaranteed if the time step size Δt does not exceed a maximum value defined by a stability criterion, expressed in terms of a diffusion coefficient, time step and spatial discretization sizes [19]. For the solution of convection-diffusion problems, the Courant-Friedrich-Levy (CFL) stability condition [2, 3] is a necessary condition for the numerical stability of explicit FD schemes. The CFL condition relates the time step size to both the propagation velocity in the advection term and the spatial discretization size. Owning to conditional stability, explicit FD schemes are difficult to apply to diffusion and convection-diffusion problems defined on domains with widely different length scales (as, for instance, thin slot, thin films, layered media) if a too long transient is desired. since the maximum time step size is constrained by the smallest length scale, a huge number of time steps may be required to guarantee stability for the entire transient analysis, which increases the CPU computation time and effort. Implicit FD schemes are unconditionally stable and thus have been widely used to avoid excessively small time-step sizes, as required by explicit schemes [4, 5]. However, in order to reduce memory requirements and computation time, combined or mixed explicit-implicit schemes have been preferred instead of fully-implicit ones. The two most widely-used mixed FD schemes are the Crank-Nicholson [19] and the alternating-direction-implicit (ADI) [4]. Although unconditionally stable, the time step size in fully-implicit or mixed FD schemes must be kept reasonably small to avoid large discretization (or dispersion) errors, which deteriorate the accuracy of the numerical solution. Hence, the computation time to achieve accurate numerical solutions may be prohibitively large even with unconditionally stable FD schemes. We also show afterwards how the accuracy of results obtained with the alternating-direction-implicit scheme deteriorates for time steps Δt much higher than its maximum value defined by the stability criterion.

In the current work, we thus propose an unconditionally stable and accurate FD scheme for the numerical solution of transient diffusion and convection-diffusion problems. The core idea is to separate the spatial and temporal dependencies of the unknown field and to expand the latter dependency as a series of orthogonal functions in the domain $[0, \infty)$. These orthogonal functions are the weighted Laguerre polynomials [8,9,12]. The resulting system of partial differential equations for spatially-variable coefficients are then solved using classical FD schemes. With the proposed Laguerre-FD scheme (L-FDM), we avoid thus any time integration and therefore we get rid of any stability criterion. The proposed L-FDM may be alternatively viewed as a combination of classical FDM with an integral transform performed in the time domain, whose orthogonal eigenfunctions are the weighted Laguerre polynomials. The Laguerre functions have been widely as spectral method to approximate different types of equations [35–41].

As will be demonstrated, the L-FDM is a marching-on-in-degree scheme instead of traditional marching-on-in-time FD schemes. Numerical stability and discretization errors are no longer affected by the time step size. Therefore, the L-FDM may be an attractive approach to tackle diffusion and convection-diffusion problems defined on domains with widely-different length scales and with a too long transient. The proposed L-FDM has been already successfully employed for the solution of transient Maxwell's equations [1]. Nevertheless, to the best knowledge of the authors, no previous work has applied the L-FDM