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Strong Convergence of the Semi-Implicit Euler Method for a Kind of Stochastic Volterra Integro-Differential Equations

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Abstract. This paper is mainly concerned with the strong convergence analysis of the semi-implicit Euler method for a kind of stochastic Volterra integro-differential equations (SVIDEs). The solvability and the mean-square boundedness of numerical solutions are presented. In view of the properties of the Itô integral, different from the known stochastic problems, it is proved that the strong convergence order of the semi-implicit Euler method is 1, although the approximation order of the Itô integral is 0.5. The theoretical results are illustrated by extensive numerical examples.

AMS subject classifications: 45D05, 60H35, 65C30, 65L20

Key words: Stochastic Volterra integro-differential equations, semi-implicit Euler methods, boundedness, convergence.

1. Introduction

Volterra integral equations (VIEs) are applied to many questions in biological, engineering, physical sciences and so on. In general, VIEs are usually difficult to solve analytically. In order to obtain the approximate solution, it is of interest to develop efficient numerical methods. H. Brunner in [5] presented a survey of methods for the numerical treatment of various types of Volterra functional equations and in paper [6] reviewed one hundred years of VIEs and dealt with some of the recent developments in the theory and numerical analysis of VIEs of the first kind. In recent years many different methods, such as homotopy perturbation method (see [19, 42]), collocation methods (see [30]), radial basis functions method (see [1, 3]), transform methods (see [40]), block pulse functions method (see [28, 29, 31]), wavelets methods (see [4]), Nystrom's method (see [2, 15]),

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Table 1: Orders of the semi-implicit Euler methods.

	ODEs	VIEs	VIDEs
Deterministic problems	1	1	1
Stochastic problems	0.5	0.5 or 1	unknown

Taylor expansion method (see [27, 38]), block-pulse functions and Taylor series method (see [32, 33]) were applied to derive numerical solutions of different kinds of integral equations.

Usually the mathematical models or equations used to describe some phenomena in the general areas of the biological, engineering, oceanographic, and physical sciences. Due to the nondeterministic nature of phenomena, the mathematical descriptions of such phenomena frequently result in random or stochastic equations. Due to the difficulty of obtaining the analytical solutions, a large number of research articles concerned with the numerical solutions of stochastic Volterra integral equations (SVIEs) have emerged, such as [9, 12, 16, 21, 34–36, 39].

Recently, mathematical modelling of real-life problems usually results in stochastic Volterra integro-differential equations (SVIDEs). Many numerical methods have been developed for solving SVIDEs. In [13, 14], Golec and Sathananthan studied the convergence of Euler-Maruyama scheme for SVIDEs under some different assumptions. Stochastic θ-methods (see [18]), collocation methods (see [26]), finite-difference methods (see [10]) and other numerical methods (see [23, 37]) for solving SVIDEs have been developed in recent years. Each of these numerical methods has its inherent advantages and disadvantages and the search for alternative, more general, easier and more accurate numerical methods is a continuous and ongoing process.

There are some research results on the convergence of the semi-implicit Euler methods for some kinds of problems. Listed in Table 1, the convergence orders for ODEs (ordinary differential eqautions), VIEs and VIDEs are all 1 (see [7, 8]), but for stochastic ODEs and VIEs, the strong orders reduce to 0.5 (see [11, 25]) since the strong approximation of the Itô integral leads to order 0.5 approximation. However, in [41], it is shown that the strong convergence order for SVIEs is 1, which needs some additional conditions on the Itô integral. Naturally, there is an interesting question: what is the convergence order of the semi-implicit Euler method for SVIDEs?

In this paper, we apply the semi-implicit Euler method to the following SVIDEs:

$$\begin{cases} dX(t) = f\left(t, X(t), \int_0^t k(t,s)X(s)ds, \int_0^t \sigma(t,s)X(s)dW(s)\right)dt, \quad t \in I := [0,T], \\ X(0) = \eta, \end{cases} \tag{1.1}$$

where $\eta \in \mathbb{R}$, the kernels $k, \sigma : D\big(:= \{(t,s) : 0 \le s \le t \le T\}\big) \to \mathbb{R}$ and the nonlinear continuous function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given, the Wiener process $W(t)$ is a one-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$ and $\mathscr{F}_t(t \ge 0)$ is an