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On Generalizations of p-Sets and their Applications

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Abstract. The p-set, which is in a simple analytic form, is well distributed in unit cubes. The well-known Weil's exponential sum theorem presents an upper bound of the exponential sum over the p-set. Based on the result, one shows that the p-set performs well in numerical integration, in compressed sensing as well as in uncertainty quantification. However, p-set is somewhat rigid since the cardinality of the p-set is a prime p and the set only depends on the prime number p. The purpose of this paper is to present generalizations of p-sets, say $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$, which is more flexible. Particularly, when a prime number p is given, we have many different choices of the new p-sets. Under the assumption that Goldbach conjecture holds, for any even number m, we present a point set, say $\mathscr{L}_{p,q}$, with cardinality $m-1$ by combining two different new p-sets, which overcomes a major bottleneck of the p-set. We also present the upper bounds of the exponential sums over $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$ and $\mathscr{L}_{p,q}$, which imply these sets have many potential applications.

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Key words: p-set, deterministic sampling, numerical integral, exponential sum.

1. Introduction

1.1. p-set

Let p be a prime number. We consider the point set

$$\mathscr{P}_{d,p}=\left\{\mathbf{x}_0,\cdots,\mathbf{x}_{p-1}\right\}\subset[0,1)^d,$$

where

$$\mathbf{x}_j=\left(\left\{\frac{j}{p}\right\},\left\{\frac{j^2}{p}\right\},\cdots,\left\{\frac{j^d}{p}\right\}\right)\in[0,1)^d,\quad j\in\mathbb{Z}_p,$$

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$\mathbb{Z}_p := \{0, 1, \cdots, p-1\}$ and $\{x\}$ is the fractional part of x for a nonnegative real number x. The point set $\mathscr{P}_{d,p}$ is called p-set and was introduced by Korobov [4] and Hua-Wang [3]. In [1], Dick introduced the development of p-set in detail. Recently, p-set attracts much attention since its advantage in numerical integration [1], in the recovery of sparse trigonometric polynomials [10] and in the uncertainty quantification [11]. Particularly, the p-set has good performance in terms of the dimension d. In [1], Dick presents a numerical integration formula based on $\mathscr{P}_{d,p}$ with showing the error bound of the formula depends only polynomially on the dimension d. In [10], Xu uses $\mathscr{P}_{d,p}$ to construct the deterministic sampling points of sparse trigonometric polynomials and shows the sampling matrix corresponding to $\mathscr{P}_{d,p}$ has the almost optimal coherence. And hence, $\mathscr{P}_{d,p}$ has a good performance for the recovery of sparse trigonometric polynomials.

1.2. Generalizations of p-set: $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$ and $\mathscr{L}_{p,q}$

The p-set is in a simple analytic form and hence it is easy to be generated by computer. However, the p-set is somewhat rigid with the point set only depending on a prime number p. If the function values at some points in p-set are not easy to be obtained, one has to change the prime number p to obtain a new point set which has the different cardinality with the previous one. Hence, in practical application, it will be better that one can choose the different point sets which have the similar property with $\mathscr{P}_{d,p}$. We next introduce a generalization of p-set.

Let

$$\mathbb{Z}_p^d := \left\{\mathbf{a} = (a_1, \cdots, a_d) : \ a_j \in \mathbb{Z}_p, \ j = 1, \cdots, d\right\}.$$

Suppose that $\mathbf{a} = (a_1, \cdots, a_d) \in \mathbb{Z}_p^d$ and $\epsilon = (\epsilon_1, \cdots, \epsilon_{d-1}) \in \{0,1\}^{d-1}$. We set

$$\mathscr{P}_{d,p}^{\mathbf{a},\epsilon} := \left\{\mathbf{x}_j^{\mathbf{a},\epsilon} : j \in \mathbb{Z}_p\right\}, \tag{1.1}$$

where

$$\mathbf{x}_j^{\mathbf{a},\epsilon} := \left(\left\{ \frac{a_1 j}{p} \right\}, \left\{ \frac{a_1' j + a_2 j^2}{p} \right\}, \cdots, \left\{ \frac{\sum_{h=1}^{d-1} a_h' j^h + a_d j^d}{p} \right\} \right) \in [0,1)^d,$$

and $a_k' = \epsilon_k a_k, k = 1, \cdots, d-1$. We call $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$ as the p-set associating with the parameter $\mathbf{a}$ and ϵ. If we take $\mathbf{a} = (1, \cdots, 1)$ and $\epsilon = (0, \cdots, 0)$, then $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$ is reduced to the classical p-set.

The p-set $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$ associating with the parameters $\mathbf{a}, \epsilon$ is more flexible. Given the prime number p, one can generate various point sets by changing the parameters $\mathbf{a}$ and ϵ with presenting an option set when the cardinality p is given.

Note that the cardinality of both $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$ and $\mathscr{P}_{d,p}$ is prime. Since the distance between adjacent prime can be very large, the cardinality of p-set does not change “smoothly". Using the set $\mathscr{P}_{d,p}^{\mathbf{a},\epsilon}$, we next present a set with the cardinality being odd number. Suppose that $m \in 2\mathbb{Z}$ is given. The Goldbach conjecture, which is one of the best-known unsolved