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Segmentation by Elastica Energy with L^1 and L^2 Curvatures: a Performance Comparison

Xuan He[1,*], Wei Zhu[1] and Xue-Cheng Tai[2]

[1] *Department of Mathematics, University of Alabama, Box 870350, Tuscaloosa, AL 35487, USA*
[2] *Department of Mathematics, Hong Kong Baptist University, Kowloon Tong Kowloon, Hong Kong*

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Abstract. In this paper, we propose an algorithm based on augmented Lagrangian method and give a performance comparison for two segmentation models that use the L^1- and L^2-Euler's elastica energy respectively as the regularization for image segmentation. To capture contour curvature more reliably, we develop novel augmented Lagrangian functionals that ensure the segmentation level set function to be signed distance functions, which avoids the reinitialization of segmentation function during the iterative process. With the proposed algorithm and with the same initial contours, we compare the performance of these two high-order segmentation models and numerically verify the different properties of the two models.

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Key words: Augmented Lagrangian method, Euler's elastica, image segmentation.

1. Introduction

Image segmentation is a typical problem in image processing, with a broad range of applications in medical image analysis, object detection, recognition, etc. It aims to partition a given image domain into several disjoint regions, each of which describes either a meaningful object or background. During the last few decades, numerous variational models have been developed for this problem. These include the snake and active contour model by Kass, Witkin, and Terzopoulus [17], the Mumford-Shah model [23], the geodesic active contour model by Caselles, Kimmel, and Sapiro [7], and the Chan-Vese model [11], to name a few. The Chan-Vese model can be regarded as a special case of the Mumford-Shah model by confining the approximation functions to be binary functions, and an attractive feature of the Chan-Vese model is its treatment of segmentation contours using level set functions [25].

*Corresponding author. *Email addresses:* `xhe12@crimson.ua.edu` (X. He), `wzhu7@ua.edu` (W. Zhu), `xuechengtai@hkbu.edu.hk` (X. C. Tai)

Recently, in [35], we considered a modification of the Chan-Vese (CV) model by employing the Euler's elastica energy as the regularization of segmentation contour. Euler's elastica energy was first seriously studied for visual perception by Mumford [22], and it has been widely utilized as a regularizer in image inpainting [1,2,10], capturing illusory contours [24,33], and image denoising [29]. Benefited from the attributes of this high-order regularizer, the modified CV model (ECV-L^2 model) presents several new features when compared with the original CV model for image segmentation (cf. [35]): 1) automatically connecting broken parts of objects; 2) capturing objects of large size while omitting small ones; 3) being more suited than the CV model for keeping elongated structures.

Later on, another variant of the Chan-Vese model was discussed in [4], where the L^1 variant of Euler's elastica was taken as the new regularization term of segmentation contours. The most remarkable feature of this new segmentation model lies in the fact that it privileges convex contours once a strong weight is imposed on the curvature term, which is supported by the theorem [21] in differential geometry that the integral of the magnitude of curvature along any closed piecewise smooth curve is greater than or equal to 2π, and the minimum value is attained only when the closed curve is convex.

To present these two variants of the Chan-Vese model, we recall the standard Euler's elastica that refers to a curve Γ that minimizes the elasticity energy

$$\mathbf{E}(\Gamma)=\int_\Gamma (a+b\kappa^2)ds \tag{1.1}$$

among all curves satisfying some boundary conditions, where κ represents the curvature of curves and $a,b>0$ are two parameters. The L^1-variant of Euler's elastica energy can be expressed as

$$\mathbf{E}(\Gamma)=\int_\Gamma (a+b|\kappa|)ds. \tag{1.2}$$

This elastica energy linearly depends on the magnitude of curvature, which helps maintain corners during the segmentation process, as discussed in the well-known segmentation with depth model by Nitzberg, Mumford, Shiota [24].

By incorporating Euler's elastica energy, those two modified Chan-Vese models can be written in the level set setting as follows:

$$\begin{aligned}\min_{\{\phi,c_1,c_2\}} E(\phi,c_1,c_2)=&\min_{\{\phi,c_1,c_2\}}\int_\Omega (f-c_1)^2H(\phi)+(f-c_2)^2(1-H(\phi))\\&+\int_\Omega\left(a+b\left|\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|}\right|^2\right)|\nabla H(\phi)|,\end{aligned} \tag{1.3a}$$

$$\begin{aligned}\min_{\{\phi,c_1,c_2\}} E(\phi,c_1,c_2)=&\min_{\{\phi,c_1,c_2\}}\int_\Omega (f-c_1)^2H(\phi)+(f-c_2)^2(1-H(\phi))\\&+\int_\Omega\left(a+b\left|\nabla\cdot\frac{\nabla\phi}{|\nabla\phi|}\right|\right)|\nabla H(\phi)|,\end{aligned} \tag{1.3b}$$