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# A Mixed Regularization Method for Ill-Posed Problems

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**Abstract.** In this paper we propose a mixed regularization method for ill-posed problems. This method combines iterative regularization methods and continuous regularization methods effectively. First it applies iterative regularization methods in which there is no continuous regularization parameter to solve the normal equation of the ill-posed problem. Then continuous regularization methods are applied to solve its residual problem. The presented mixed regularization algorithm is a general framework. Any iterative regularization method and continuous regularization method can be combined together to construct a mixed regularization method. Our theoretical analysis shows that the new mixed regularization method is with optimal order of error estimation and can reach the optimal order under a much wider range of the regularization parameter than the continuous regularization method such as Tikhobov regularization. Moreover, the new mixed regularization method can reduce the sensitivity of the regularization parameter and improve the solution of continuous regularization methods or iterative regularization methods. This advantage is helpful when the optimal regularization parameter is hard to choose. The numerical computations illustrate the effectiveness of our new mixed regularization method.

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**Key words**: Ill-posedness, continuous regularization, iterative regularization, mixed regularization, optimal order.

## 1. Introduction

Suppose that $T : X \to Y$ is a bounded linear operator, here $X$ and $Y$ are two Hilbert spaces. The inverse problem is that $y \in Y$ is known and we seek $x \in X$ such that $Tx = y$. Ill-posedness always appears in inverse problems if $T$ is compact, i.e., the solution $x$

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may not satisfy existence, uniqueness or continuity. Even if we can ensure the existence and uniqueness of the solution $x$ in some sense, the non-continuity still leads to difficulty in computations. The non-continuity means that the solution $x$ is very sensitive to any perturbation of the right-hand side $y$, i.e., small perturbations of $y$ can produce arbitrarily large perturbations of the solution $x$. The perturbation of $y$ usually represents the noises in the data. The method to overcome the ill-posedness is known as the regularization [3, 14, 42]. Generally speaking, the regularization methods can be classified into two categories: continuous regularization methods and iterative regularization methods. In this article we try to combine the two types of regularization methods together and we name it the mixed regularization method.

The typical continuous regularization methods include the truncated singular value decomposition (TSVD) method and Tikhonov regularization method. Among these methods, Tikhonov regularization method is the widely used method. It is proposed firstly by Tikhonov [37, 38] in 1963 and then is applied in solving ill-posed problems [39, 40]. Other methods include the stationary and non-stationary iterated Tikhonov method and so on [13]. In the continuous regularization methods, one important step is the choice of the regularization parameter. If the parameter is too small, the ill-posedness of the original problem can not be overcome effectively and the error from the ill-posedness is dominated. On the contrary, if the parameter is too large, the consistency between the original problem and the regularized problem become large and the error from this factor is dominated. Therefore, it is crucial to balance these two kinds of errors. Many methods on how to choose the optimal parameter are investigated, for example, the discrepancy principle [18,27,33,41,44], the L-curve criterion [15,16], the generalized cross-validation [1,6,43], and so on. However, how to choose a optimal regularization parameter effectively is still worth studying in solving real problems when we have little knowledge about the exact solution.

Iterative regularization methods are another type of frequently-used regularization methods. In iterative regularization methods, the stopping index is regularization parameter and there is no continuous regularization parameter. For well-posed problems, the iterative solution usually convergences to the exact solution as the number of iterations increases. However, for ill-posed problems, there exists a phenomenon called semi-convergence, i.e., the iterative solution converges to the exact solution in first several iterations, but it goes away from the exact solution after a certain step. Thus, some stopping rules must be used so that the iteration can stop at certain iteration which is closest to the exact solution. The popular iterative regularization methods include the Landweber method [2, 9, 12, 21, 26, 30, 32], the conjugate gradient (CG) method [5, 10, 11, 25, 28, 29] and so on [14].

In iterative regularization methods, the number of iterations or the stopping index is a regularization parameter and this parameter is on the set of natural number. In contrast, the regularization parameter in continuous regularization methods is on the set of real numbers. Therefore, in iterative regularization methods, the parameter (i.e., the number of iterations) can not be chosen as precisely as that in continuous regularization methods. For example, the iterative solution usually does not approximate the exact solution