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A Two-Steps Method Based on Plane Wave for Nonhomogeneous Helmholtz Equations in Inhomogeneous Media

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Abstract. In this paper we are concerned with numerical methods for nonhomogeneous Helmholtz equations in inhomogeneous media. We try to extend the plane wave method, which has been widely used to the discretization of homogeneous Helmholtz equations with constant wave numbers, to solving nonhomogeneous Helmholtz equations with piecewise constant wave numbers. To this end, we propose a combination between the plane wave discontinuous Galerkin (PWDG) method and the high order element discontinuous Galerkin (HODG) method for the underlying Helmholtz equations. In this composite methods, we need only to solve a series of local Helmholtz equations by the HODG method and solve a locally homogeneous Helmholtz equation with element-wise constant wave number by the PWDG method. In particular, we first consider non-matching grids so that the generated approximations have better performance for the current case. Numerical experiments show that the proposed methods are very effective to the tested Helmholtz equations.

AMS subject classifications: 65N30, 65N55

Key words: Nonhomogeneous Helmholtz equation, inhomogeneous media, plane wave method, discontinuous Galerkin method, errors, preconditioner.

1. Introduction

Consider the Helmholtz equations in inhomogeneous media

$$\begin{cases} -\Delta u - \kappa^2 u = f & \text{in } \Omega, \\ \nabla u \cdot \boldsymbol{n} + i\kappa u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded connected Lipschitz domain in $\mathbb{R}^2$, $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$ and κ is the wave number defined by $\kappa(\boldsymbol{x}) = \omega/c(\boldsymbol{x})$, with $c(\boldsymbol{x})$ being the wave speed. Here, the wave speed $c(\boldsymbol{x})$ is not a constant function on Ω, i.e., the involved media is inhomogeneous.

Helmholtz equation is the basic model in sound propagation. It is a very important topic to design a high accuracy method for Helmholtz equations with large wave numbers, such that the so called "wave number pollution" can be reduced. In recent years, many interesting methods for the discretization of Helmholtz equations with large wave numbers have been proposed, for example (but not all), the first-order system least-squares (FOS-LS) method [22], the ultra weak variational formulation (UWVF) [3], the plane wave least squares (PWLS) methods [17, 18, 23], the plane wave discontinuous Galerkin (P-WDG) methods [14, 16], the method of fundamental solutions [2, 5], the plane wave method with Lagrange multipliers [9], the variational theory of complex rays [24], the high order element discontinuous Galerkin method (HODG) [8, 11], local discontinuous Galerkin method (LDG) [12], hybridizable discontinuous Galerkin method (HDG) [4, 15] and the discontinuous Petrov-Galerkin method [7, 28]. All these methods are superior to the standard linear finite element method.

It is well known that the plane wave methods can generate higher accuracy approximations than the other methods for solving the Helmholtz equation with large (piecewise constant) wave number. In addition, we found that, when the same preconditioner is used to solve the Helmholtz systems derived by the plane wave method and the other methods, the preconditioned GMRES iteration has obviously faster convergence in the case of the plane wave method (see numerical results reported in Section 5). Unfortunately, the plane wave methods can be directly applied only to the discretization of homogeneous Helmholtz equations. In fact, the plane wave approximate solutions of the nonhomogeneous Helmholtz equations usually have low accuracies only (see [14]). We can naturally ask a question: is it possible to solve nonhomogeneous Helmholtz equations with variable wave numbers by the plane wave method (combining some other methods)? Of course, we hope that the new method still possesses better performances than the other methods for solving the considered Helmholtz equations.

In this paper we consider the case that $c(\boldsymbol{x})$ is a piecewise constant function on Ω, which was considered in [19], and we design a combination between the PWDG method and HODG method to solve the Helmholtz equations (1.1). The basic idea is to transform a nonhomogeneous Helmholtz equation with piecewise constant wave number into a locally homogeneous Helmholtz equation with element-wise constant wave number and a series of local nonhomogeneous Helmholtz equations. Then the homogeneous Helmholtz equation with element-wise constant wave number can be solved by the PWDG method, and all the local Helmholtz equations can be solved in parallel by the (local) HODG method. A similar idea was proposed in [18] for the discretization of nonhomogeneous time-harmonic Maxwell's equations with constant wave numbers, but the domains defining the local equations mentioned above are different from that in [18]. In particular, we choose nonmatching grids in the proposed method so that the generated approximations have better performance for the current case (with piecewise constant wave number).