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Finite Element and Discontinuous Galerkin Methods with Perfect Matched Layers for American Options

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Abstract. This paper is devoted to the American option pricing problem governed by the Black-Scholes equation. The existence of an optimal exercise policy makes the problem a free boundary value problem of a parabolic equation on an unbounded domain. The optimal exercise boundary satisfies a nonlinear Volterra integral equation and is solved by a high-order collocation method based on graded meshes. This free boundary is then deformed to a fixed boundary by the front-fixing transformation. The boundary condition at infinity (due to the fact that the underlying asset's price could be arbitrarily large in theory), is treated by the perfectly matched layer technique. Finally, the resulting initial-boundary value problems for the option price and some of the Greeks on a bounded rectangular space-time domain are solved by a finite element method. In particular, for Delta, one of the Greeks, we propose a discontinuous Galerkin method to treat the discontinuity in its initial condition. Convergence results for these two methods are analyzed and several numerical simulations are provided to verify these theoretical results.

AMS subject classifications: 91G60, 91G80, 65M6
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1. Introduction

European options allow the owner to exercise the option only on the specified expiration dates, while American options can be exercised on or before the expiration dates. In a celebrated paper, Black and Scholes [9] formulated a model (the Black-Scholes model) that governs the option's price over time, they also gave a closed-form solution for European options. Unlike European options, there is no closed-form solution available for American options. Kim [25] found that the American option pricing problem can be regarded as a European option plus an early exercise premium. Based on this idea, Carr,

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Jarrow and Myneni [12] reformulated this problem as a free boundary (optimal exercise boundary) value problem, and they wrote down the solution in a form that depends on the optimal exercise boundary, which, however, is hard to evaluate. In practise, there are several known approximation formulas to the American option value, see [4, 23]. However, these results only hold for small or large, which are not applicable to moderate time to expire. Therefore, numerical methods are highly involved in the evaluation of American options. The binomial method introduced by Cox, Ross, and Rubinstein [17] is a simple and elegant method, and its convergence has been shown to be of order one in the temporal direction by Amin and Khanna [3]. In [10, 22, 34], several finite difference schemes were designed for pricing American options, while Han and Wu [19] introduced a fast numerical method based on a nonlocal boundary condition for the same problem. Some other developed numerical methods such as finite element method have also been proposed for the Black-Scholes model, and we refer the reader to [20, 24, 26, 32, 36] and references therein.

In the present paper, we concentrate on the evaluation of the price as well as the Greeks for an American option. The Greeks, as important risk measures in practice, are some certain partial derivatives of the option value with respect to different variables or parameters. For the precise definitions of these Greeks, see Subsection 3.3. As we shall see, there are two major difficulties in solving the American option problem due to the two boundaries:

(i) The optimal exercise boundary. There exists an optimal exercise policy for the American option, which makes the problem under consideration to be a free boundary value problem. This boundary satisfies a nonlinear Volterra integral equation, for which it is not easy to get a good approximation of the solution.

(ii) The boundary at infinity. An asymptotic boundary condition is prescribed at infinity due to the fact that the underlying stock price could be arbitrarily large in theory. Since we cannot adopt numerical methods directly to the unbounded domain, how to truncate the domain and to control the truncated error are key issues in designing the numerical scheme.

The goal of this work is to solve the above two problems using the graded mesh collocation method (for the former) and the perfectly matched layer technique (for the later), and to propose a finite element method (FEM) as a numerical solver for the option value and some of the Greeks. As to Delta, one of the Greeks, there is another problem due to the discontinuity in its initial value, and we adopt a discontinuous Galerkin (DG) method to treat this problem. Previous works on computing American option price using finite element and finite difference methods could be found in [19, 20]. Holmes and Yang [20] also provided a method for computing the Delta via a numerical evaluation of an integral.

The problem (i) is well studied in the existing literature. Cox *et al.* [17] showed that the optimal exercise boundary satisfies a nonlinear Volterra integral equation, and Ma *et al.* [31] used a high-order collocation method to solve this integral equation. We shall adopt the treatment in [31] to solve the free boundary, and then use the front-fixing technique in [20] to transform the curved boundary to a fixed line.