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Deferred Correction Methods for Forward Backward Stochastic Differential Equations

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Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. The deferred correction (DC) method is a classical method for solving ordinary differential equations; one of its key features is to iteratively use lower order numerical methods so that high-order numerical scheme can be obtained. The main advantage of the DC approach is its simplicity and robustness. In this paper, the DC idea will be adopted to solve forward backward stochastic differential equations (FBSDEs) which have practical importance in many applications. Noted that it is difficult to design high-order and relatively "clean" numerical schemes for FBSDEs due to the involvement of randomness and the coupling of the FSDEs and BSDEs. This paper will describe how to use the simplest Euler method in each DC step–leading to simple computational complexity–to achieve high order rate of convergence.

AMS subject classifications: 60H35, 65C20, 60H10

Key words: Deferred correction method, forward backward stochastic differential equations, Euler method, high-order scheme.

1. Introduction

This work is concerned with the forward-backward stochastic differential equations (FBSDEs) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(s, X_s, Y_s, Z_s)\mathrm{d}t + \int_0^t \sigma(s, X_s, Y_s, Z_s)\mathrm{d}W_s, & \text{(FSDE)} \\ Y_t = \xi + \displaystyle\int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, & \text{(BSDE)} \end{cases} \tag{1.1}$$

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where $t \in [0,T]$ with $T > 0$ being the deterministic terminal time; $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered complete probability space with $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ being the natural filtration of the standard m-dimensional Brownian motion $W = (W_t)_{0\leq t\leq T}$; $X_0 \in \mathscr{F}_0$ is the initial condition for the forward stochastic differential equation (FSDE); $\xi \in \mathscr{F}_T$ is the terminal condition for the backward stochastic differential equation (BSDE); $b : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m} \to \mathbb{R}^d$ and $\sigma : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m} \to \mathbb{R}^{d\times m}$ are referred to the drift and diffusion coefficients, respectively; $f : \Omega\times[0.T]\times\mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m} \to \mathbb{R}^q$ is called the generator of BSDE, and $(X_t, Y_t, Z_t) : [0,T]\times\Omega \to \mathbb{R}^d\times\mathbb{R}^q\times\mathbb{R}^{q\times m}$ is the unknown.

We point out that $b(\cdot,x,y,z)$, $\sigma(\cdot,x,y,z)$, and $f(\cdot,x,y,z)$ are all $\mathcal{F}_t$-adapted for any fixed numbers x, y, and z, and that the two stochastic integrals with respect to W_s are of the Itô type. A triple (X_t, Y_t, Z_t) is called an L^2-adapted solution of the FBSDEs (1.1) if it is $\mathscr{F}_t$-adapted, square integrable, and satisfies (1.1). The FBSDEs (1.1) are called *decoupled* if b and σ are independent of Y_t and Z_t.

Our interest is to design numerical schemes which can effectively find numerical solutions of the FBSDEs (1.1). Great efforts have been made to the numerical solutions of BSDEs, see, e.g., [1,3,8,17,19,20]. However, for the fully coupled FBSDEs (1.1), there exist only few numerical studies and satisfactory results [6,11,18]. In fact, it is very difficult to design high-order and relatively "clean" numerical schemes for FBSDEs due to the fully coupling of the FSDEs and BSDEs. We mention the work in [18], where a class of multi-step type schemes are proposed, which turns out to be effective in obtaining relatively high accurate solutions for (1.1).

In this paper, we will approximate the solutions of the FBSDEs (1.1) based on the classical deferred correction (DC) method. The DC approach was first introduced in [16] to solve ordinary differential equations (ODEs). Its main idea is to use some low-order and simple schemes iteratively to achieve a high-order scheme. The terminology of *deffered correction* was formally appeared in [15], while its convergence theory for ODEs was established by Hairer [9]. In the past few decades, DC methods have been successfully applied to solve ODEs, see, e.g., [5,10,14] as well as partial differential equations (PDEs), see, e.g., [4,7]. Our main task in this work is to design highly accurate numerical methods for the fully coupled FBSDEs based on the DC approach. More precisely, we will describe how to use the simplest Euler method in each iteration step–leading to lower overall computational complexity–to end up with high-order of convergence. Moreover, numerical experiments will demonstrate that the resulting DC-based scheme is highly accurate and stable.

The rest of the paper is organized as follows. Section 2 provides some relevant preliminaries, while Section 3 presents the general framework of the deferred correction methods for FBSDEs. More detailed construction of the DC-based algorithms for decoupled and coupled FBSDEs are discussed in Sections 4. In Section 5, several numerical experiments are presented to demonstrate the effectiveness of the proposed scheme. Some concluding remarks will be given in Section 6.

Some notation to be used: