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Lacunary Interpolation by Fractal Splines with Variable Scaling Parameters

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Abstract. For a prescribed set of lacunary data $\{(x_\nu, f_\nu, f''_\nu) : \nu = 0, 1, \ldots, N\}$ with equally spaced knot sequence in the unit interval, we show the existence of a family of fractal splines $S_b^\alpha \in \mathcal{C}^3[0,1]$ satisfying $S_b^\alpha(x_\nu) = f_\nu$, $(S_b^\alpha)^{(2)}(x_\nu) = f''_\nu$ for $\nu = 0, 1, \ldots, N$ and suitable boundary conditions. To this end, the unique quintic spline introduced by A. Meir and A. Sharma [SIAM J. Numer. Anal. 10(3) 1973, pp. 433-442] is generalized by using fractal functions with variable scaling parameters. The presence of scaling parameters that add extra "degrees of freedom", self-referentiality of the interpolant, and "fractality" of the third derivative of the interpolant are additional features in the fractal version, which may be advantageous in applications. If the lacunary data is generated from a function Φ satisfying certain smoothness condition, then for suitable choices of scaling factors, the corresponding fractal spline S_b^α satisfies $\|\Phi^r - (S_b^\alpha)^{(r)}\|_\infty \to 0$ for $0 \le r \le 3$, as the number of partition points increases.

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Key words: lacunary interpolation, fractal interpolation function, variable scaling parameters, Meir-Sharma quintic spline, convergence.

1. Introduction

Through his land mark papers [1,3], Barnsley commenced the study of fractal interpolation by using the framework of Iterated Function System (IFS). Since then, many researchers have explored the technique and earnestly attempted to generalize the notion of Fractal Interpolation Function (FIF) in many different ways. As a new type of interpolant, FIF enjoys more advantages than the classical interpolation methods, which are based on polynomials, trigonometric functions, rational functions, and splines. To put in a nutshell, the main advantages of FIFs over traditional nonrecursive interpolants

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are: (i) they provide a method to render non-smooth approximants (ii) by suitable selection of parameters of the underlying IFS, FIFs can be made smooth and these smooth FIFs include traditional interpolants as special cases (iii) interpolation scheme produced by fractal functions can have local or global dependence on data points, depending on the choice of scaling factors (iv) interpolants possess self-referentiality (v) the interpolant or a certain derivative of it has a non-integer box-counting dimension, which can be controlled by scaling factors.

If the IFS is chosen appropriately in terms of a prescribed continuous function f, then the notion of fractal interpolation can be used to produce a family of fractal functions $\{f^\alpha\}$, which includes f as a very special case. This was first observed by Barnsley and later popularized by Navascués through a series of papers (see, for instance, [9, 10]). The free parameter α, which is a suitable vector in the Euclidean space, enables us to preserve or modify properties of the original function f. In particular, each element of this class can be made to preserve smoothness of the original. The methodology is so versatile and the corresponding notion of α-fractal function acts as a medium by which the theory of fractal interpolation overlaps and interacts quite fruitfully with many other fields of mathematics. In the perspective of numerical analysis, the notion of α-fractal function is used to generalize some well-known traditional interpolation techniques such as Hermite interpolation and splines [5, 11], but not explored in the area of lacunary interpolation. Furthermore, in much of the researches in fractal functions, the free parameters termed scaling factors, which have decisive influence on the properties of the "perturbed function", are restricted to be constants. Deriving principal influence from these facts, the present article targets to invite fractal functions with variable scalings to the field of lacunary interpolation.

To achieve the intended goal, a family of fractal splines is constructed as fractal perturbation (having function scaling parameters) of a quintic spline with $\mathcal{C}^3$-continuity introduced in [8]. This perturbation process allows one to replace the unicity of the traditional quintic spline that solves the lacunary interpolation problem with unicity up to a particular choice of scaling vector. This has practical advantage: the lack of unicity opens up the possibility of choosing an interpolant that fit a certain application best, for instance, in a problem that involves both approximation and optimization. Further, in contrast to the traditional quintic spline $S \in \mathcal{C}^3(I)$, the perturbed function $S_b^\alpha \in \mathcal{C}^3(I)$ has the property that its third derivative $(S_b^\alpha)^{(3)}$ may reveal, in general, non-smooth or fractal characteristic which can be quantified in terms of Minkowski dimension [6]. The fractal characteristic of the interpolant may be explored in various nonlinear and nonequilibrium phenomena. On the other hand, for suitable choice of scaling functions, the fractal spline introduced herein has same approximation properties as that of its classical counterpart. Thus, the current article may be considered as a humble attempt to (i) re-investigate [8] using fractal interpolation, a methodology which is not yet very familiar to the "traditional" numerical analysts, (ii) reiterate the ubiquity of fractal function by taking lacunary interpolation - a field where fractal splines are not yet explored - as a medium, and (iii) pronounce that approximation by fractal functions can provide more flexibility, which may be exploited in various practical applications.