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Analysis of a Streamline-Diffusion Finite Element Method on Bakhvalov-Shishkin Mesh for Singularly Perturbed Problem

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Abstract. In this paper, a bilinear Streamline-Diffusion finite element method on Bakhvalov-Shishkin mesh for singularly perturbed convection – diffusion problem is analyzed. The method is shown to be convergent uniformly in the perturbation parameter ϵ provided only that $\epsilon \leq N^{-1}$. An $\mathcal{O}(N^{-2}(\ln N)^{1/2})$ convergent rate in a discrete streamline-diffusion norm is established under certain regularity assumptions. Finally, through numerical experiments, we verified the theoretical results.

AMS subject classifications: 65N15; 65N30

Key words: singularly perturbed problem, Streamline-Diffusion finite element method, Bakhvalov-Shishkin mesh, error estimate.

1. Introduction

In this paper, we consider a Streamline-Diffusion finite element method (SDFEM) for the singularly perturbed boundary value problem

$$\begin{aligned}&Lu := -\epsilon\Delta u + b\cdot\nabla u + cu = f \quad \text{on} \quad \Omega=(0,1)^2,\\ &u=0 \quad \text{on} \quad \partial\Omega,\end{aligned} \tag{1.1}$$

where $0<\epsilon\ll 1$ is a small positive parameter, b, c and f are sufficiently smooth functions satisfying

$$b(x,y)=(b_1(x,y),b_2(x,y))\geq(\beta_1,\beta_2)>(0,0), \quad \forall (x,y)\in\bar{\Omega}, \tag{1.2a}$$

$$c(x,y)\geq 0, \quad c(x,y)-\frac{1}{2}\text{div}b(x,y)\geq c_0>0, \quad \forall (x,y)\in\bar{\Omega}, \tag{1.2b}$$

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where β_1, β_2 and c_0 are some constants. These hypotheses ensure that (1.1) has a unique solution in $H_0^1(\Omega)\bigcap H^2(\Omega)$ for all $f \in L^2(\Omega)$. Note that for sufficiently small ϵ, the other hypotheses imply that (1.2b) can always be ensured by the simple change of variable $v(x,y) = e^{-\sigma x}u(x,y)$ where σ is chosen suitably. With the above assumptions, the solution of (1.1) typically has boundary layers of width $\mathcal{O}(\epsilon \ln \frac{1}{\epsilon})$ at the outflow boundary $x = 1$ and $y = 1$.

For small values of ϵ, standard Galerkin discretisation for (1.1) exhibits spurious oscillations and fails to catch the rapid change of the solution in boundary layers, see the numerical results in [15]. Many methods have been developed to overcome the numerical difficulty caused by the boundary layers.

One of the most successful methods is the use of layer-adapted meshes. Provided that some information on the structure of the layers was available, a piecewise uniform Shishkin mesh(S-mesh) could be chosen a priori, see [1, 3]. Linß [8, 9] introduced Bakhvalov-Shishkin mesh(B-S-mesh) which is a modification of S-mesh by using a uniform coarse mesh and a graded fine mesh with Shishkin's simple choice of the transition point. The optimal convergence order $\mathcal{O}(N^{-1})$ on a B-S-mesh had been proved, while on S-mesh it was only convergent of $\mathcal{O}(N^{-1}\ln N)$. Zhang [5] investigated the superconvergence of order $\mathcal{O}(N^{-2}(\ln N)^2)$ in a discrete ϵ-weighted energy norm on a S-mesh.

A powerful method for stabilising convection-diffusion problems is the streamline-diffusion finite element method which was proposed by Hughes and Brooks [16]. This method was known to provide good stability properties and high accuracy in boundary layers. The convergence properties of the SDFEM had been widely studied[3,10-13]. In [13], the error between the SDFEM solution and the interpolation of the solution of (1.1) on S-mesh was of order $\mathcal{O}(N^{-3/2}\ln N)$ in the streamline-diffusion norm(SD norm). In [10], a more careful analysis was performed by using interpolation error identities of Lin, and this error was improved to $\mathcal{O}(N^{-2}(\ln N)^2)$. In order to achieve estimates for the interpolation error in SD norm, Stynes and Tobiska [10] firstly introduced the discrete streamline-diffusion norm, and estimated an error bound of order $\mathcal{O}(N^{-2}(\ln N)^2)$ on S-mesh.

Here we shall analyze a SDFEM on B-S-mesh, and it will give more accurate results than on S-mesh. There are three main results in this paper. First, the interpolation error in discrete SD norm is presented to be convergent of $\mathcal{O}(N^{-2})$. Second, the error between the solution of the discrete problem and the interpolation of the solution of the continuous problem is shown to be bounded in discrete SD norm by $\mathcal{O}(N^{-2}(\ln N)^{1/2})$, uniformly in ϵ. Third, we prove that the error between the solution of the discrete problem and the solution of the continuous problem itself can be estimated in discrete SD norm by $\mathcal{O}(N^{-2}(\ln N)^{1/2})$.

An outline of the paper is as follows. In Section 2 we describe the B-S-mesh and the SDFEM. A decomposition of the solution u and some important preliminaries to the analysis are presented in Section 3, and in Section 4 we analyze the convergence properties of the method. In order to validate our theoretical results, numerical results are presented in Section 5. We end in Section 6 with some concluding remarks.