Notes on New Error Bounds for Linear Complementarity Problems of Nekrasov Matrices, $B$-Nekrasov Matrices and $QN$-Matrices

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Abstract. In this paper, we give new error bounds for linear complementarity problems when the matrices involved are Nekrasov matrices, $B$-Nekrasov matrices and $QN$-matrices, respectively. It is proved that the obtained bounds are better than those of Li et al. (New error bounds for linear complementarity problems of Nekrasov matrices and $B$-Nekrasov matrices, Numer. Algor., 74 (2017), pp. 997–1009) and Gao et al. (New error bounds for linear complementarity problems of $QN$-matrices, Numer. Algor., 77 (2018), pp. 229–242) in some cases, respectively.

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1. Introduction

The linear complementarity problem is to find a vector $x \in \mathbb{R}^n$ such that
\begin{equation}
x \geq 0, \quad Ax + q \geq 0, \quad (Ax + q)^T x = 0,
\end{equation}
or to show that no such vector $x$ exists, where $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. We abbreviate this problem by LCP$(A, q)$. Many problems can be posed in the form (1.1). For instance, problems in linear and quadratic programming, the problem of finding a Nash equilibrium point of a bimatrix game or some free boundary problems of fluid mechanics (see [1-3,29-32]). It is well known that a real square matrix $A$ is called
a $P$-matrix if all its principal minors are positive. $A$ is a $P$-matrix if and only if the LCP($A, q$) has a unique solution $x^*$ for any $q \in \mathbb{R}^n$ (see [2]).

Some error bounds for LCPs of $P$-matrices are derived (see [4-7]). Particularly, when the matrix $A$ of the LCP($A, q$) (1.1) is a $P$-matrix, Chen and Xiang [4] derived the following error bound

$$\| x - x^* \|_{\infty} \leq \max_{d \in [0,1]^n} \| (I - D + DA)^{-1} \| \| r(x) \|_{\infty},$$

where $x^*$ is the solution of LCP($A, q$), $r(x) := \min(x, Ax + q)$, $D = \text{diag}(d_i)$ with $0 \leq d_i \leq 1$, and the min operator denotes the componentwise minimum of two vectors.

When the involved matrix belongs to a subclass of $P$-matrices, such as $H$-matrices with positive diagonals, $B$-matrices, $DB$-matrices, $SB$-matrices, $MB$-matrices, $B^S$-matrices and weakly chained diagonally dominant $B$-matrices, many error bounds for the LCPs (1.1) are achieved in the literature (see [8-17, 21-24, 27-28]). In [15,16], error bounds for linear complementarity problem with Nekrasov matrices and $B$-Nekrasov matrices are presented respectively. Recently, Li et al. [21] provided new error bounds for LCPs($A, q$) associated with Nekrasov matrices and $B$-Nekrasov matrices and Gao et al. [23] presented a new error bound for LCP($A, q$) involved with a $QN$-matrix, which are only dependent on the entries of the matrix $A$.

In this paper, we find new error bounds of linear complementarity problem when the involved matrices are Nekrasov matrices, $B$-Nekrasov matrices and $QN$-matrices. It is proved that the given bounds improve corresponding bounds of [21], for Nekrasov matrices and $B$-Nekrasov matrices, and [23], for $QN$-matrix in some cases. In particular, when the positive diagonal entries of the related matrix $A$ are located in an interval $(0,1]$, it is proved that the new bound is generally sharper than that of Remark 2.4 in [9]. Related numerical examples show that the new bounds are tighter than those derived recently.

2. A new error bound for LCPs of Nekrasov matrices

Let us first introduce some basic notations and some classes of matrices. We denote $N := \{1, \ldots, n\}$ and by $e := (1, \ldots, 1)^T$ the unit column vector of $n$ elements. A $Z$-matrix is a matrix whose off-diagonal elements are nonpositive and a nonsingular $M$-matrix is a $Z$-matrix with nonnegative inverse. Given a real matrix $A$, its comparison matrix $\langle A \rangle = (\bar{a}_{ij}) \in \mathbb{R}^{n \times n}$ defined by setting $\bar{a}_{ii} = |a_{ii}|$ and $\bar{a}_{ij} = -|a_{ij}|$, $i \neq j$, $i, j \in N$. If $\langle A \rangle$ is an $M$-matrix, then $A$ is called an $H$-matrix. We say that a matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant by rows ($SDD$) if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all $i, j \in N$.

In what follows, we recall the definition of Nekrasov matrices [18-19] and prepare several fundamental lemmas.

**Definition 2.1.** Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}, n \geq 2$, with $a_{ii} \neq 0, i \in N$. We say that $A$ is a Nekrasov matrix if, for all $i \in N$,

$$|a_{ii}| > h_i(A),$$