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A Hybridized Weak Galerkin Finite Element Method for Incompressible Stokes Equations

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Abstract. In this paper, a hybridized weak Galerkin (HWG) finite element method is proposed for solving incompressible Stokes equations. The finite element space of the proposed method is constructed simply by piecewise polynomials. The optimal convergence order can be achieved for velocity function both in L^2 norm and H^1 norm, pressure function in H^1 norm. Finally, a Schur complement is employed to reduce the degree of freedom in discrete problem. Numerical examples are presented to demonstrate the effectiveness of the hybridized weak Galerkin finite element method.

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1. Introduction

The incompressible Stokes equations describe the relationship of the velocity, pressure, temperature and density of incompressible fluids. There are various problems can be modeled by the Stokes equations, for example, distribution of static pressure, multiphase flow driven by surface tension. Coupling with supplemental equations, the Stokes equations can model flow in highly heterogeneous media appropriately [1, 17, 20], and fluids in complex media with multiphysics, which have important impacts on environmental and industrial problems.

In this paper, we assume that the incompressible Stokes equations are defined in Ω, a polygonal or polyhedral domain in $\mathbb{R}^d$ (for $d=2,3$) with Lipschitz-continuous boundary $\partial\Omega$. We denote the equations as follows

$$-\mu\Delta\mathbf{u}+\nabla p=\mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$
$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega, \tag{1.1b}$$
$$\mathbf{u}=\mathbf{g} \quad \text{on } \partial\Omega, \tag{1.1c}$$

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where μ represents the fluid viscosity, $\mathbf{u}$ and p denote velocity and pressure, respectively. Function $\mathbf{f}$ is a momentum source term. The boundary function $\mathbf{g}$ satisfies the compatible condition

$$\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\, ds = 0,$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.

We assume that p has zero average with the purpose of guaranteeing the uniqueness of the pressure function, i.e.,

$$\int_{\Omega} p\, d\Omega = 0.$$

Throughout this paper, we use standard definitions of Lebesgue and Sobolev spaces: $L^2(\Omega)$, $H^1(\Omega)$ and $[L^2(\Omega)]^d$. For simplicity, we assume $\mu = 1$ and $\mathbf{g} = \mathbf{0}$. Then the classical variational formulations for equations (1.1a)-(1.1c) are to find $(\mathbf{u};p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ such that

$$(\nabla\mathbf{u},\nabla\mathbf{v}) - (\nabla\cdot\mathbf{v},p) = (\mathbf{f},\mathbf{v}), \tag{1.2a}$$
$$(\nabla\cdot\mathbf{u},q) = 0, \tag{1.2b}$$

for all $(\mathbf{v};q) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$, where

$$[H_0^1(\Omega)]^d = \left\{\mathbf{v}\in[H^1(\Omega)]^d : \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\right\},$$
$$L_0^2(\Omega) = \left\{q\in L^2(\Omega) : \int_\Omega q dx = 0\right\}.$$

Obviously, the Stokes problems (1.1a)-(1.1c) can rewrite the following variational formulations: finding $(\mathbf{u};p) \in [H_0^1(\Omega)]^d \times (L_0^2(\Omega)\cap H^1(\Omega))$ satisfies

$$(\nabla\mathbf{u},\nabla\mathbf{v}) + (\nabla p,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \tag{1.3a}$$
$$(\nabla q,\mathbf{u}) = 0, \tag{1.3b}$$

for all $(\mathbf{v};q) \in [H_0^1(\Omega)]^d \times (L_0^2(\Omega)\cap H^1(\Omega))$. In [18], we take the parameters $m = 0$ and $r = 1$ in the Theorem 5.4. So we can obtain L^2 regularity. The pressure function $p \in H^1(\Omega)$ has the gradient function ∇p and the above variational forms (1.2a)-(1.2b) have a unique solution. Therefore, we can get the solvability of variational equations (1.3a)-(1.3b) directly.

Many numerical methods have been developed for solving the Stokes equations. The finite difference method (see, e.g., [28,30]) is a popular one. We can easily get the discrete systems of the equations with this method. The low-order mixed finite element method [19] for the Stokes equation is a stabilized approach, the global upper limit and local lower limit of the *a posteriori* error analysis and finite element discretization error of the method are given. The finite volume method [29] for the two-dimensional Stokes equations also gives a corresponding nonconforming finite element method, making the systems of linear equations equivalent. Numerical experiments show that the velocity converges for some special