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High Order Padé Schemes for Nonlinear Wave Propagation Problems

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Abstract

Split-step Padé method and split-step fourier method are applied to the higher-order nonlinear Schrödinger equation. It is proved that a combination of Padé scheme and spectral method is the most effective method, which has a spectral-like resolution and good stability nature. In particular, we propose an unconditional stable implicit Padé scheme to solve odd order nonlinear equations. Numerical results demonstrate the excellent performance of Padé schemes for high order nonlinear equations.

Keywords: Split-step Padé method; split-step fourier method; Padé scheme; spectral method; nonlinear equations.
Mathematics subject classification: 65M06

1. Introduction

It is well known that pseudo-spectral methods are commonly used for obtaining accurate solutions to initial-boundary value problems. However, especially with second (or higher) order derivatives in space, the corresponding differential matrices have large spurious eigenvalues (leading, for example, to severe time step restrictions in case of explicit time-stepping methods). Therefore, split-step Fourier scheme [10] or implicit spectral method [14] is often used to relax the time step restriction.

However, it is found that similar numerical results and higher computational efficiency are obtained by using Padé scheme for nonlinear subproblem and spectral method for linear subproblem rather than a direct use of split-step spectral scheme. For odd order nonlinear equations, we also propose an implicit Padé scheme with an artificial viscosity term, which is unconditional stable, to replace filtered implicit spectral method.

This paper is organized as follows. In Section 2, we give a brief review of Padé scheme. In Section 3, We apply three different combinations of operator splitting method for higher-order nonlinear Schrödinger equation (HNLS). In Section 4, we propose an implicit Padé scheme to $K(i,i,i)$ equation and $Q(2,2,2)$ equation. Numerical experiments are also carried out. Concluding remarks are made in Section 5.

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2. Brief review of Padé scheme

For the computation region $[a,b]$, choose a global spatial mesh size $h = \Delta x$ with $h = (b-a)/M$, where M is an even positive integer and the time step $k = \Delta t$. The grid points and time steps are

$$x_m = a + mh, \quad m = 0, 1, \cdots, M,$$
$$t_n = nk, \quad n = 0, 1, 2, \cdots,$$

and denote the approximation of $u(x_m, t_n)$ by u_m^n. If we denote

$$\begin{aligned}
&E^\alpha u_m^n = u_{m+\alpha}^n, \quad \mu = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}}),\\
&\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}, \quad \overline{\mu} = \frac{1}{2}(E + E^{-1}),\\
&\overline{\delta} = \frac{1}{2}(E - E^{-1}), \quad \delta^2 = E - 2 + E^{-1},\\
&\mu\delta = \frac{1}{2}(E - E^{-1}) = \overline{\delta}, \quad \overline{\mu}\overline{\delta} = \frac{1}{4}(E^2 - E^{-2}).
\end{aligned}$$

Padé scheme is to approximate the space derivatives ($f_x = f'$, $f_{xx} = f''$, etc) with a truncation error of $\mathcal{O}(h^4)$, which can be written as operator form as follows [6,7]:

$$(1 + 2\alpha\overline{\mu})f_m' = \frac{1}{h}(b1 \cdot \overline{\mu}\overline{\delta} + a1 \cdot \overline{\delta})f_m, \tag{2.1}$$

$$(1 + 2\alpha\overline{\mu})f_m'' = \frac{1}{h^2}(b2 \cdot \overline{\delta}^2 + a2 \cdot \delta^2)f_m, \tag{2.2}$$

$$(1 + 2\alpha\overline{\mu})f_m''' = \frac{1}{h^3}(b3 \cdot \overline{\delta}^3 + a3 \cdot \overline{\delta}\delta^2)f_m, \tag{2.3}$$

$$(1 + 2\alpha\overline{\mu})f_m'''' = \frac{1}{h^4}[b4 \cdot (\frac{2}{3}\overline{\delta}^2\delta^2 + \frac{1}{3}\delta^4) + a4 \cdot \delta^4]f_m, \tag{2.4}$$

$$(1 + 2\alpha\overline{\mu})f_m''''' = \frac{1}{h^5}[b5 \cdot (\frac{2}{3}\overline{\delta}^3\delta^2 + \frac{1}{3}\overline{\delta}\delta^4) + a5 \cdot \overline{\delta}\delta^4]f_m. \tag{2.5}$$

According to Taylor series expansion, we only need

$$\begin{aligned}
&a1 = \frac{2}{3}(\alpha + 2), \quad b1 = \frac{1}{3}(4\alpha - 1), \quad a2 = \frac{4}{3}(1 - \alpha),\\
&b2 = \frac{1}{3}(-1 + 10\alpha), \quad a3 = 2, \quad b3 = 2\alpha - 1,\\
&a4 = 2(1 - \alpha), \quad b4 = 4\alpha - 1, \quad a5 = 3, \quad b5 = 2\alpha - 2,
\end{aligned} \tag{2.6}$$

to reach a truncation error of order $\mathcal{O}(h^4)$. In order to simplify computations, we often set one of coefficients $b_J = 0$ (sometimes J is the highest order of derivatives of the equation) to get the value of α. It has been proved in [13] that using the above rule to choose the value of α is more accurate and efficient than directly setting $\alpha = 0$. Padé scheme is a high accurate and stable method for solving nonlinear problems.