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On Vector Helmholtz Equation with a Coupling Boundary Condition

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Abstract

The Helmholtz equation is sometimes supplemented by conditions that include the specification of the boundary value of the divergence of the unknown. In this paper, we study the vector Helmholtz problem in domains of both $C^{1,1}$ and Lipschitz. We establish a rigorous variational analysis such as equivalence, existence and uniqueness. And we propose finite element approximations based on the uncoupled solutions. Finally we present a convergence analysis and error estimates.

Keywords: Helmholtz equation; coupling boundary condition; variational formulation; uncoupled finite element solution; convergence analysis.
Mathematics subject classification: 65Z05, 65N12, 65N30

1. Introduction

In some physical problems, such as the determination of the electric field vector in the electromagnetic fields within closed perfectly conducting cavities that may contain dielectric and/or magnetic materials, see [1, 2], or the vector potential used to represent either 3-dimensional velocity field in incompressible flows [4] or the solenoidal velocity component in transonic potential flows for a compressible fluid [5] as well as the vorticity in 3-dimensional viscous in compressible flows [6], it is necessary to consider a vector field

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governed by a vector Helmholtz equation supplemented with boundary conditions of the following type:

$$\begin{cases} (a) & -\Delta \boldsymbol{u} + \gamma \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega; \\ (b) & \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{n} \quad \text{on } \Gamma; \\ (c) & \nabla \cdot \boldsymbol{u} = d \quad \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $\Omega \subset \Re^D$, $D = 2$ or 3, Γ is the boundary of Ω, γ is a positive constant, $\boldsymbol{f}$ is a given field in Ω, $\boldsymbol{a} \times \boldsymbol{n}$ and d are, respectively, a given field and a given function on Γ, $\boldsymbol{n}$ is the unit outer normal vector to Γ.

It is easily seen that the presence of the divergence in the boundary condition (1.1c) implies a coupling between the components of unknown $\boldsymbol{u}$, which generally prevents solving problem (1.1) as a system of two or three independent scalar equations for the Cartesian components of the unknown $\boldsymbol{u}$. Hence it is worthwhile to investigate possible strategies aimed at reducing the coupled vector problem to a set of independent scalar equations for the numerical solution of (1.1). The purpose of this paper is to extend all of the results for Poisson problem in [3,10] to Helmholtz problem (1.1).

The organization of this paper is as follows. In Section 2, we give the equivalent variational formulations of the system (1.1) for $C^{1,1}$ and Lipschitz domains, respectively, and establish the existence and uniqueness results for the corresponding formulations. Then, we derive the uncoupled solutions and propose the finite element approximations in Section 3. The convergence analysis and the error estimates will be presented in Section 4.

2. Variational formulation

2.1. Problem in a $C^{1,1}$ domain

Assume that Ω is $C^{1,1}$ simply connected. We recall the following spaces defined in [3,10]:

$$\boldsymbol{H}^1_N(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \,|\, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma\}, \tag{2.1}$$

$$\boldsymbol{H}^1_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \,|\, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma\}, \tag{2.2}$$

$$\boldsymbol{H}^1_a(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \,|\, \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{n} \quad \text{on } \Gamma\}, \tag{2.3}$$

$$\boldsymbol{H}^1_{T,a}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}^1_a(\Omega) \,|\, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma\}. \tag{2.4}$$

The above spaces are equipped with the norm $\|\cdot\|_a$ associated with the following inner product:

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) + \gamma(\boldsymbol{u}, \boldsymbol{v}).$$

The following assumptions are needed:

$$\begin{cases} (a) & \boldsymbol{f} \in \boldsymbol{H}^{-1}_N(\nabla\cdot, \Omega), \\ (b) & \boldsymbol{a} \times \boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma), \\ (c) & d \in H^{-1/2}(\Gamma), \end{cases} \tag{2.5}$$