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The Mortar Element Method with Lagrange Multipliers for Stokes Problem

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Abstract

In this paper, we propose a mortar element method with Lagrange multiplier for incompressible Stokes problem, i.e., the matching constraints of velocity on mortar edges are expressed in terms of Lagrange multipliers. We also present P_1 nonconforming element attached to the subdomains. By proving inf-sup condition, we derive optimal error estimates for velocity and pressure. Moreover, we obtain satisfactory approximation for normal derivatives of the velocity across the interfaces.

Keywords: Lagrange multipliers; inf-sup condition; Stokes problem.
Mathematics subject classification: 65N55, 65N22, 65N30

1. Introduction

During the last ten years, more and more people devoted to studying incompatible grids. It allows great flexibility for generating locally structured/globally unstructured meshes in complex geometries. Meanwhile, the mortar element method was introduced by Bernardi, Maday and Patera in [1]. It is a nonconforming domain decomposition method with non-overlapping subdomains. The meshes on different subdomains need not align across subdomains interfaces and the matching of discretizations on adjacent subdomains is only enforced weakly. Recently, the method has been studied extensively and many results have been obtained, see, e.g., [2–5, 8].

Viewed as a generalization of nonconforming methods, mortar element method with Lagrange multipliers removes the continuity across the subdomains interfaces and expends in introducing a Lagrange multiplier. This technique was used by Raviat and Thomas in [10] who constructed finite element approximation of elliptic problem based on a primal hybrid variational principle. Then, Belgacem further developed the method and proved the error bounds in [5]. Now, we apply it to Stokes problem by expressing the matching constraints of velocity on subdomains interfaces (i.e., the mortar edges) in terms of Lagrange multipliers. For each triangulation of subdomains, the P_1 nonconforming element is equipped. By checking the corresponding inf-sup condition, we prove the problem has a unique solution and obtain error estimates.

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This paper is organized as follows: in Section 2, we review Stokes problem and analyze the variational formulations for geometrical conforming decompositions. Section 3 presents a finite element discretization and proves inf-sup condition. The last section gives the evaluation of the error. Throughout this paper, we denote by C a universal constant independent of the mesh size and level, but whose values can differ from place to place.

2. Model problem and domain decomposition

We consider the following model problem

$$\left\{\begin{array}{ll} -\Delta \mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ div\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \end{array}\right. \tag{2.1}$$

where Ω is bounded convex polygonal domain in $\mathscr{R}^2$, $\mathbf{u}$ represents the velocity of fluid, p is pressure and $\mathbf{f}$ is external force. Define

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega), \int_\Omega q\,dx = 0 \right\}.$$

The mixed variational formulation of (2.1) is to find $\mathbf{u} \in (H_0^1(\Omega))^2$ and $p \in L_0^2(\Omega)$ such that

$$\left\{\begin{array}{ll} \bar{a}(\mathbf{u},\mathbf{v}) + \bar{b}(\mathbf{v},p) = <\mathbf{f},\mathbf{v}>, & \forall \mathbf{v} \in (H_0^1(\Omega))^2, \\ \bar{b}(\mathbf{u},q) = 0, & \forall q \in L_0^2(\Omega), \end{array}\right. \tag{2.2}$$

where the bilinear formulations $\bar{a}(\cdot,\cdot)$ defined over $(H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$, $\bar{b}(\cdot,\cdot)$ defined over $(H_0^1(\Omega))^2 \times L_0^2(\Omega)$ and the duality inner product $<\cdot,\cdot>$ are given respectively by

$$\begin{aligned} \bar{a}(\mathbf{u},\mathbf{v}) &:= \int_\Omega \nabla\mathbf{u}\nabla\mathbf{v}\,dx, \\ \bar{b}(\mathbf{v},q) &:= -\int_\Omega div\,\mathbf{v}q\,dx, \\ <\mathbf{f},\mathbf{v}> &:= \int_{\boldsymbol{\Omega}} \mathbf{f}\mathbf{v}\,\mathbf{dx}. \end{aligned}$$

It is well-known that the bilinear form $\bar{b}(\mathbf{v},q)$ satisfies the inf-sup condition, i.e., there exists a positive constant $\bar{\beta}$ for any $q \in L_0^2(\Omega)$ such that

$$\sup_{\mathbf{v}\in(H_0^1(\Omega))^2} \frac{\bar{b}(\mathbf{v},q)}{\|\mathbf{v}\|_{(H^1(\Omega))^2}} \geq \bar{\beta}\|q\|_0.$$