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Stability of T. Chan's Preconditioner from Numerical Range[†]

Cheman Cheng[1], Xiaoqing Jin[1,*] and Vaikuong Sin[2]

[1] *Department of Mathematics, University of Macau, Macao.*
[2] *Department of Electromechanical Engineering, University of Macau, Macao.*

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Abstract. A matrix is said to be stable if the real parts of all the eigenvalues are negative. In this paper, for any matrix A_n, we discuss the stability properties of T. Chan's preconditioner $c_U(A_n)$ from the viewpoint of the numerical range. An application in numerical ODEs is also given.

Key words: T. Chan's preconditioner; stability; numerical range; boundary value method.

AMS subject classifications: 65F10, 65F15, 65L05, 65FN22

1 Introduction

T. Chan [9] proposed a circulant preconditioner for Toeplitz matrices in 1988. R. Chan, Jin and Yeung [6] showed that T. Chan's preconditioner can be defined not only for Toeplitz matrices but also for general matrices. Given a unitary matrix $U \in \mathbb{C}^{n\times n}$, define

$$\mathcal{M}_U \equiv \{U^*\Lambda_n U \mid \Lambda_n \text{ is any } n\text{-by-}n \text{ diagonal matrix}\}. \tag{1}$$

For any matrix $A_n \in \mathbb{C}^{n\times n}$, T. Chan's preconditioner $c_U(A_n) \in \mathcal{M}_U$ is defined to the minimizer of

$$\|c_U(A_n) - A_n\| = \min_{W_n\in\mathcal{M}_U} \|W_n - A_n\|,$$

where $\|\cdot\|$ is the Frobenius norm. Let F denote the Fourier matrix whose entries are given by:

$$(F)_{j,k} = \frac{1}{\sqrt{n}} e^{2\pi \mathbf{i}(j-1)(k-1)/n}, \qquad \mathbf{i} \equiv \sqrt{-1}, \quad 1 \le j,k \le n. \tag{2}$$

When $U = F$ in (1), $\mathcal{M}_U$ is the set of all circulant matrices [11]. It is proved that T. Chan's circulant preconditioner is a good preconditioner for solving a large class of linear systems, see, e.g., [4,5,7,9,16,17].

*Correspondence to: Xiaoqing Jin, Department of Mathematics, University of Macau, Macao. Email: xqjin@umac.mo

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In this paper, we will study some stability properties of T. Chan's preconditioner from the viewpoint of numerical range. The stability property is essential in many applications, including control theory and dynamical systems [1]. We first introduce the following definition.

Definition 1.1. A matrix is said to be stable if the real parts of all the eigenvalues are negative.

We now briefly review some important results. For any matrix $E \in \mathbb{C}^{n\times n}$, let $\delta(E)$ denote a diagonal matrix whose diagonal is equal to the diagonal of E. For T. Chan's preconditioner, we have the following lemma, see [6, 15, 22].

Lemma 1.1. *Let $A_n \in \mathbb{C}^{n\times n}$ and $c_U(A_n)$ be T. Chan's preconditioner. Then*

(i) $c_U(A_n)$ is uniquely determined by A_n and is given by

$$c_U(A_n) \equiv U^*\delta(UA_nU^*)U.$$

(ii) If A_n is Hermitian, then $c_U(A_n)$ is also Hermitian. Moreover, we have

$$\min_j \lambda_j(A_n) \le \min_j \lambda_j\big(c_U(A_n)\big) \le \max_j \lambda_j\big(c_U(A_n)\big) \le \max_j \lambda_j(A_n),$$

where $\lambda_j(E)$ is the j-th eigenvalue of E.

From Lemma 1.1 (ii), it is easy to see that if A_n is Hermitain and stable, then so is $c_U(A_n)$. In [19], Jin et al. showed that if A_n is normal and stable, then $c_U(A_n)$ is also normal and stable. The result is further generalized in [3]. It is proved that if A_n is $*$-congruent to a stable diagonal matrix, i.e., $A_n = Q^*DQ$ where Q is a nonsingular matrix and D is a stable diagonal matrix, then $c_U(A_n)$ is stable. Recently, by noting that any matrix A_n can be written as

$$A_n = H + \mathbf{i}K,$$

where

$$H = \frac{1}{2}(A_n + A_n^*) \quad \text{and} \quad K = \frac{1}{2\mathbf{i}}(A_n - A_n^*)$$

are Hermitian, Cheng and Jin proved the following result:

Lemma 1.2. ([10]) *Let $A_n \in \mathbb{C}^{n\times n}$ and suppose that $A_n = H + \mathbf{i}K$ where H and K are Hermitian. Then T. Chan's preconditioner $c_U(A_n)$ is stable for any unitary matrix $U \in \mathbb{C}^{n\times n}$ if and only if H is negative definite.*

It is a well-known fact that $*$-congruence does not change the inertia of a Hermitian matrix. Furthermore, for Hermitian matrices H and K with H nonsingular, H and K are simultaneously diagonalizable by $*$-congruence if and only if $H^{-1}K$ has real eigenvalues and is diagonalizable [13, p.229]. Suppose now that H is positive definite. Then $H^{-1}K$ is similar to $H^{-1/2}KH^{-1/2}$ which is Hermitian. Therefore, H and K are simultaneously diagonalizable by $*$-congruence. Of course, the same conclusion holds when H is negative definite. Thus, by using Lemma 1.2, one can show that the condition in [3], i.e., A_n is $*$-congruent to a stable diagonal matrix, is actually a necessary and sufficient condition for $c_U(A_n)$ to be stable for all unitary U. However, the condition in Lemma 1.2 is much simpler.

Another result concerning the stability of $c_U(A)$ is the following:

Lemma 1.3. ([10]) *Let $A_n \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $c_U(A_n)$ is stable if and only if*

$$\mathrm{Re}[\mathrm{tr}(A_n)] < 0,$$

where $\mathrm{Re}[\cdot]$ *denotes the real part of a complex number and* $\mathrm{tr}(\cdot)$ *denotes the trace of a matrix.*