NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **16** No. 1, pp. 74-82
February 2007

Mixed Method for Compressible Miscible Displacement with Dispersion in Porous Media

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Received March 30, 2005; Accepted (in revised version) January 19, 2006

Abstract. Compressible miscible displacement of one fluid by another in porous media is modelled by a nonlinear parabolic system. A finite element procedure is introduced to approximate the concentration of one fluid and the pressure of the mixture. The concentration is treated by a Galerkin method while the pressure is treated by a parabolic mixed finite element method. The effect of dispersion, which is neglected in [1], is considered. Optimal order estimates in L^2 are derived for the errors in the approximate solutions.

Key words: Compressible miscible displacement; mixed finite element method; dispersion; error estimate.

AMS subject classifications: 65N30

1 Introduction

Miscible displacement of one compressible fluid by another in a porous medium is modeled by a nonlinear parabolic system [1, 4]

$$d(c)\frac{\partial p}{\partial t} + \nabla \cdot u = d(c)\frac{\partial p}{\partial t} - \nabla \cdot (a(c)\nabla p) = q, \tag{1a}$$

$$\phi(x)\frac{\partial c}{\partial t} + b(c)\frac{\partial p}{\partial t} + u \cdot \nabla c - \nabla \cdot (D(u)\nabla c) = (\hat{c} - c)q, \tag{1b}$$

where c denotes the volumetric concentration of one of the two components of the fluid ($c = c_1 = 1 - c_2$), and p denotes its pressure. The coefficients $a(c)$, $b(c)$, $d(c)$, $\phi(x)$ (porosity of the rock) are assumed bounded below positively and $a(c)$, $b(c)$, $d(c) \in C^1$; q is the external volumetric flow rate; $\hat{c}$ is the concentration of the external flow, which is specified at points where injection ($q > 0$) takes place, or assumed to be equal to c at production points; u is the Darcy velocity of the fluid satisfying

$$u = -a(c)\nabla p, \tag{2}$$

$D(u)$ combines the effects of molecular diffusion and dispersion [5], defined by

$$D = \phi\{d_m I + |u|(d_l E(u) + d_t E^{\perp}(u))\} \tag{3}$$

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where $E(u) = [u_k u_l/|u|^2]$ is an 2×2 matrix representing orthogonal projection along the velocity vector and $E^{\perp}(u) = I - E(u)$ is its orthogonal complement. $D(u)$ is a positive definite matrix since the effect of molecular diffusion is much greater than that of dispersion. In addition, the reservoir Ω will be taken to be of unit thickness and be identified with a bounded domain in R^2.

We shall also assume that no flow occurs across the boundary:

$$u \cdot \nu = 0 \qquad on\ \partial\Omega, \tag{4a}$$

$$(D\nabla c - cu)\cdot \nu = 0 \quad on\ \partial\Omega, \tag{4b}$$

where ν is the outer normal to $\partial\Omega$. The initial conditions are

$$p(x,0) = p_0(x) \quad x \in \Omega, \tag{5a}$$

$$c(x,0) = c_0(x) \quad x \in \Omega. \tag{5b}$$

Douglas [1] introduced a mixed finite element procedure for the same problem while dispersion was neglected such that $D = \phi(x)d_m I$. Cheng [8] introduced a Galerkin procedure with dispersion on rectangle element and derived optimal error estimates. Wang and Cheng [9] considered the Galerkin method with dispersion to another similar problem on quasi-regular element, and derived nearly optimal error estimates. In this paper, a mixed finite element procedure on quasi-regular element is introduced with dispersion so that $D = D(u)$ (see (3)). The analysis of the procedure is based on [1] while different test functions are selected and two projections are introduced to derive the optimal error estimates in L^2.

2 Formulation of the mixed finite element procedure

It is well known that physical transport dominates the diffusive effects in realistic examples of compressible miscible displacement. Thus it is more important to obtain good approximate velocities than to achieve high accuracy in pressure. This motivates the use of mixed method in the calculation of the pressure and velocity.

Firstly, the weak form for the parabolic system (1a), (1b) and (2) is given by

$$\left(\phi\frac{\partial c}{\partial t}, z\right) + \left(b(c)\frac{\partial p}{\partial t}, z\right) + (u\cdot\nabla c, z) + (D(u)\nabla c, \nabla z) = ((\hat{c}-c)q, z) \quad z \in H^1(\Omega), t \in J, \tag{6a}$$

$$\left(d(c)\frac{\partial p}{\partial t}, w\right) + (\nabla\cdot u, w) = (q, w) \qquad w \in L^2(\Omega), t \in J, \tag{6b}$$

$$(\alpha(c)u, v) - (\nabla\cdot v, p) = 0 \qquad v \in V, t \in J, \tag{6c}$$

where $V = \{v \in H(div;\Omega) : v\cdot\nu = 0\ on\ \partial\Omega\}$, $\alpha(c) = a(c)^{-1}$ and $J = (0,T]$.

Let $h = (h_c, h_p)$, where h_c and h_p are positive. Let M_h denote a standard finite element space whose elements diameters are bounded by h_c. Assume that M_h is associated with a quasi-regular polygonalization of Ω and piecewise-polynomial functions of some fixed degree greater or equal to l. As a result, all standard inverse relations hold on M_h, which will be used frequently in our analysis. Then let $V_h \times W_h$ be a Raviart-Thomas [6] space of index at least k associated with quasi-regular triangulation or quadrilateralization (or a mixture of the two) of Ω such that the elements diameters are bounded by h_p. If the approximations for the pressure, concentration and velocity are denoted by $c_h \in M_h$, $p_h \in W_h$ and $u_h \in V_h$ respectively, then they are defined