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New Conservative Schemes for Regularized Long Wave Equation†

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Abstract. In this paper, two finite difference schemes are presented for initial-boundary value problems of Regularized Long-Wave(RLW) equation. They all have the advantages that there are discrete energies which are conserved. Convergence and stability of difference solutions with order $\mathcal{O}(h^2+\tau^2)$ are proved in the energy norm. Numerical experiment results demonstrate the effectiveness of the proposed schemes.

Key words: RLW equation; difference scheme; conservation; convergence.
AMS subject classifications: 65M06, 65M99

1 Introduction

The Regularized Long-Wave (RLW) equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^2\partial t}=0 \tag{1}$$

is first proposed by Peregrine in [1] to describe the development of an undular bore, which describes wave motion to the same order of approximation as the KDV equation. The initial value and boundary value (IVBV) conditions for the equation are

$$u|_{t=0}=u_0(x), \tag{2}$$

$$u|_{x=X_L}=u|_{x=X_R}=0, \tag{3}$$

where $u_0(x)$ is a smooth function. Assume $u_x,\rho_x\to 0$ when $x\to X_L$ or $x\to X_R$. Then the IVBV problem (1)–(3) possesses the following conservative quantities [6]:

$$E(t)=\frac{1}{2}(||u||^2+||u_x||^2)=\frac{1}{2}(||u_0||^2+||u_{0x}||^2)=E(0), \tag{4}$$

$$Q(t)=\frac{1}{2}\int_{X_L}^{X_R}u(x,t)dx=\frac{1}{2}\int_{X_L}^{X_R}u_0(x)dx=Q(0), \tag{5}$$

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where $E(0)$ is a constant which depends only on the initial value.

Numerical methods for the RLW equation have been considered in several papers. A simple finite difference method for the RLW equation was first proposed by Peregrine in [1]. Finite element schemes and spectral methods were studied in [2, 5, 7-9, 11]. In [3, 4], Eilbeck and McGuire proposed a two level scheme and a three level scheme, but the two schemes are not conservative. It is well known that the conservative schemes perform better than the nonconservative ones. In [10], Chang proposed a two level conservative scheme, but iterations are needed, and thus require more CPU time. In [12], Zhang proposed an interesting linearized finite difference conservative scheme. In this paper, we will design two new conservative schemes which are convergent with convergence rate of two in a discrete L_∞ norm. One scheme is a three level linearized scheme which involves a parameter η, $0 \le \eta \le 1$, and another is a two level non-linearized one.

The paper is organized as follows. Two new conservative schemes are given in Section 2. The prior estimates for numerical solutions are established in Section 3. In Section 4, convergence and stability for the new schemes are studied. Numerical experiments are reported in the last section.

2 The difference scheme and its conservation laws

In this section, we describe two difference schemes for the RLW equations. As usual, the following notations are used:

$$x_j = X_L + jh, \quad t_n = n\tau, \quad j = 0, 1, \cdots, J, \quad n = 0, 1, \cdots, [T/\tau] = N,$$

where $h = \frac{X_R - X_L}{J}$ and τ denote the spatial and temporal mesh sizes respectively, $u_j^n \equiv u(x_j, t_n)$, and $U_j^n \sim u(x_j, t_n)$. Let

$$\begin{aligned}
&(V_j^n)_x = \frac{V_{j+1}^n - V_j^n}{h}, \quad (V_j^n)_{\overline{x}} = \frac{V_j^n - V_{j-1}^n}{h}, \quad (V_j^n)_{\hat{x}} = \frac{1}{2}((V_j^n)_x + (V_j^n)_{\overline{x}}),\\
&(V_j^n)_t = \frac{V_j^{n+1} - V_j^n}{\tau}, \quad (V_j^n)_{\overline{t}} = \frac{V_j^n - V_j^{n-1}}{\tau}, \quad (V_j^n)_{\hat{t}} = \frac{1}{2}((V_j^n)_t + (V_j^n)_{\overline{t}}),\\
&(U, V) = h\sum_{j=1}^{J-1} U_j \overline{V}_j, \quad ||V||^2 = (V, V), \quad ||V||_\infty = \max_{1 \le j \le J} |V_j|.
\end{aligned}$$

Throughout the paper, C denotes a general positive constant, which may have different values in different occurrences.

Now, we consider the following finite difference schemes for the problem (1)-(3):

Scheme A. Three levels:

$$\begin{aligned}
&(U_j^n)_{\hat{t}} + \frac{h^2}{12}(U_j^n)_{x\bar{x}\hat{t}} + \frac{1-\eta}{2}(U_j^{n+1} + U_j^{n-1})_{\hat{x}} + \eta(U_j^n)_{\hat{x}} - (U_j^n)_{x\bar{x}\hat{t}} + \frac{1}{6}[U_j^n(U_j^{n+1} + U_j^{n-1})_{\hat{x}}\\
&\qquad + (U_j^n(U_j^{n+1} + U_j^{n-1}))_{\hat{x}}] = 0, \ j = 1, 2, \cdots, J-1; \ n = 1, 2, \cdots, N, \qquad (6)\\
&U_j^0 = u_0(X_L + jh), \quad j = 0, 1, 2, \cdots, J, \qquad (7)\\
&U_0^n = U_J^n = 0, \quad n = 0, 1, 2, \cdots, N, \qquad (8)
\end{aligned}$$

where $\eta \in [0, 1]$. It should be pointed out that we need another suitable two level scheme (such as Scheme B as follows) to compute U^1.