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THE NOTE ON MATRIX-VALUED RATIONAL INTERPOLATION*

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Abstract *In* [3], *a kind of matrix-valued rational interpolants* (MRIs) *in the form of* $R_n(x) = M(x)/D(x)$ *with the divisibility condition* $D(x) \mid \|M(x)\|^2$, *was defined, and the characterization theorem and uniqueness theorem for* MRIs *were proved. However this divisibility condition is found not necessary in some cases. In this paper, we remove this restricted condition, define the generalized matrix-valued rational interpolants* (GMRIs) *and establish the characterization theorem and uniqueness theorem for* GMRIs. *One can see that the characterization theorem and uniqueness theorem for* MRIs *are the special cases of those for* GMRIs. *Moreover, by defining a kind of inner product, we succeed in unifying the Samelson inverses for a vector and a matrix.*

Key words *generalized matrix-valued rational interpolants , reduced matrix-valued rational interpolants, uniqueness.*

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1 Introduction

Suppose $\boldsymbol{X}_n = \{x_i \in \boldsymbol{R}^1 : i = 0, 1, \cdots, n\}$ is a set of real points distinct from each other and $\{\boldsymbol{A}_i \in \boldsymbol{C}^{d_1 \times d_2} : i = 0, 1, \cdots, n\}$ is the corresponding set of $d_1 \times d_2$ finite valued complex matrices. The matrix-valued rational interpolation is to find the matrix function

$$\boldsymbol{R}_n(x) = \boldsymbol{M}(x)/D(x) \tag{1.1}$$

such that

$$\boldsymbol{R}_n(x_i) = \boldsymbol{M}(x_i)/D(x_i), \qquad i = 0, 1, \cdots, n, \tag{1.2}$$

where $\boldsymbol{M}(x) = (h_{ij}(x))_{d_1 \times d_2}$ is a $d_1 \times d_2$ real (or complex) polynomial matrix, $i = 1, 2, \cdots, d_1; j = 1, 2, \cdots, d_2$, and $D(x)$ is a real polynomial. $\partial \boldsymbol{M} := \max\limits_{1 \le i \le d_1, 1 \le j \le d_2} \{\partial h_{ij}\}$. According to [3], the Samelson inverse of a $d_1 \times d_2$ matrix $\boldsymbol{A} = (a_{ij})_{d_1 \times d_2}$ is defined as

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^*/\|\boldsymbol{A}\|^2, \quad \boldsymbol{A} \neq \boldsymbol{O}, \tag{1.3}$$

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where $\boldsymbol{A}^*$ denotes the complex conjugate of matrix $\boldsymbol{A}$ and $\|\boldsymbol{A}\|^2 = \mathrm{tr}(\boldsymbol{A}(\boldsymbol{A}^*)^T) = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2} |a_{ij}|^2$, where $(\boldsymbol{A}^*)^T$ denotes the complex conjugate transpose of matrix $\boldsymbol{A}$ and $|a_{ij}|$ is the modulus of a_{ij}. It is not difficult to derive $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.

By means of the Samelson inverse of matrix defined in (1.3), the following Thiele type matrix-valued continued fraction was constructed (see in [3],[4]).

Define

$$\boldsymbol{B}[x_i] := \boldsymbol{A}_i,\ i = 0, 1, \cdots, n, \tag{1.4}$$

$$\boldsymbol{B}[x_0, x_i] := \frac{x_i - x_0}{\boldsymbol{B}[x_i] - \boldsymbol{B}[x_0]}, \qquad i = 1, 2, \cdots, n, \tag{1.5}$$

and for $k = 1, 2, \cdots, n-1;\ i = k+1, k+2, \cdots, n$,

$$\boldsymbol{B}[x_0, x_1, \cdots, x_k, x_i] := \frac{x_i - x_k}{\boldsymbol{B}[x_0, x_1, \cdots, x_{k-1}, x_i] - \boldsymbol{B}[x_0, x_1, \cdots, x_{k-1}, x_k]}. \tag{1.6}$$

For $k = 0, 1, \cdots, n$, define

$$\boldsymbol{B}_k := \boldsymbol{B}[x_0, x_1, \cdots, x_{k-1}, x_k]. \tag{1.7}$$

The resulting construct is

$$\boldsymbol{R}_n(x) = \boldsymbol{B}_0 + \frac{x - x_0}{\boldsymbol{B}_1} + \frac{x - x_1}{\boldsymbol{B}_2} + \cdots + \frac{x - x_{n-1}}{\boldsymbol{B}_n}. \tag{1.8}$$

By the tail-to-head rationalization, we see that $\boldsymbol{R}_n(x)$ takes the form $\boldsymbol{R}_n(x) = \boldsymbol{M}(x)/D(x)$ defined in (1.1).

Theorem 1.1[3] If all matrices $\boldsymbol{B}[x_0, x_1, \cdots, x_k, x_i](k = 1, 2, \cdots, n-1;\ i = k+1, k+2, \cdots, n)$ exist and do not vanish, then the matrix-valued rational fraction $\boldsymbol{R}_n(x) = \boldsymbol{M}(x)/D(x)$ defined in (1.4)-(1.8) satisfies $\boldsymbol{R}_n(x_i) = \boldsymbol{M}(x_i)/D(x_i) = \boldsymbol{A}_i,\ i = 0, 1, \cdots, n$, and $D(x) \mid \|\boldsymbol{M}(x)\|^2$.

Definition 1.1[3] A matrix-valued rational fraction $\boldsymbol{R}_n(x) = \boldsymbol{M}(x)/D(x)$, as defined in (1.4)-(1.8), is called a matrix-valued rational interpolant (MRI) if

(1) $D(x)$ is real and $D(x) \mid \|\boldsymbol{M}(x)\|^2$;

(2) $\boldsymbol{R}_n(x_i) = \boldsymbol{A}_i,\ i = 0, 1, \cdots, n$.

AMRI, $\boldsymbol{R}_n(x) = \boldsymbol{M}(x)/D(x)$, is said to be reduced if $D(x)$ is real and all possible real polynomial common factors of $\boldsymbol{M}(x)$ and $D(x)$ have been removed, consistently with $D(x) \mid \|\boldsymbol{M}(x)\|^2$.

Definition 1.2[3] A matrix-valued rational fraction $\boldsymbol{R}_n(x) = \boldsymbol{M}(x)/D(x)$, as defined in (1.1), is said to be of type $[l/m]$ if $\partial D = M,\ \partial h_{ij} \le l,\ i = 1, 2, \cdots, d_1;\ j = 1, 2, \cdots, d_2$, and there exists some $(i_0, j_0)(1 \le i_0 \le d_1,\ 1 \le j_0 \le d_2)$, such that $\partial h_{i_0 j_0} = l$.

Characterization Theorem[3] Let a MRI is of the form

$$\boldsymbol{R}_n(x) = \frac{\boldsymbol{M}(x)}{D(x)} = \boldsymbol{B}_0 + \frac{x - x_0}{\boldsymbol{B}_1} + \frac{x - x_1}{\boldsymbol{B}_2} + \cdots + \frac{x - x_{n-1}}{\boldsymbol{B}_n}. \tag{1.9}$$

If n is even, $\boldsymbol{R}_n(x)$ is normally of type $[n/n]$. If n is odd, $\boldsymbol{R}_n(x)$ is normally of type $[n/n-1]$.

Based on the definition of MRI and the characterization theorem, the uniqueness theorem was established as follows.