English Series | NUMERICAL MATHEMATICS A Journal of Chinese Universities | Vol.14, No.3 Aug. 2005

COMPONENTWISE CONDITION NUMBERS FOR GENERALIZED MATRIX INVERSION AND LINEAR LEAST SQUARES*

Wei Yimin(魏益民) Xu Wei (许威) Qiao Sanzheng(乔三正) Diao Huaian(刁怀安)

Abstract *We present componentwise condition numbers for the problems of Moore-Penrose generalized matrix inversion and linear least squares. Also, the condition numbers for these condition numbers are given.*

Key words *Condition numbers, componentwise analysis, generalized matrix inverses, linear least squares.*

AMS(2000)subject classifications 15A12, 65F20, 65F35

1 Introduction

Condition number is a measurement of the sensitivity of a problem to the perturbation in its inputs. In general, consider a function $f(x)$. Suppose that the input x is perturbed by Δx. The condition number κ for the problem $f(x)$ quantifies the magnification of the relative errors caused by the perturbation. Specifically, κ satisfies

$$\frac{|f(x+\Delta x)-f(x)|}{|f(x)|} \le \kappa \frac{|\Delta x|}{|x|}.$$

Assuming $|\Delta x| \le \epsilon\, |x|$, we can define the condition number

$$\kappa = \lim_{\epsilon \to 0^+} \sup_{|\Delta x| \le \epsilon\, |x|} \frac{|f(x+\Delta x)-f(x)|}{\epsilon\, |f(x)|}.$$

In the problem of inverting a nonsingular matrix A, the condition number

$$\kappa(A) = \|A\|\, \|A^{-1}\|$$

* The first author is supported by the NSF of China under grant 10471027 and Shanghai Education Commission.
Received: Sep. 1, 2004.

represents the ratio between the relative errors in A and its inverse:

$$\frac{\|(A+\Delta A)^{-1}-A^{-1}\|}{\|A^{-1}\|} \leq \frac{\kappa(A)}{1-\kappa(A)\|\Delta A\|/\|A\|}\frac{\|\Delta A\|}{\|A\|},$$

assuming the perturbation ΔA is small relative to A [4]. In this paper, $\|\cdot\|$ denotes the 2-norm. The condition number for solving a nonsingular system of linear equations $Ax=b$ is also $\kappa(A)=\|A\|\,\|A^{-1}\|$ in that

$$\frac{\|(A+\Delta A)^{-1}(b+\Delta b)-A^{-1}b\|}{\|A^{-1}b\|} \leq \kappa(A)\left(\frac{\|\Delta A\|}{\|A\|}+\frac{\|\Delta b\|}{\|b\|}\right)+O(\epsilon^2),$$

for ΔA and Δb such that $\|\Delta A\| \leq \epsilon\,\|A\|$, $\|\Delta b\| \leq \epsilon\,\|b\|$, and $A+\Delta A$ is nonsingular [4].

In the general case when A can be rectangular or rank-deficient, the Moore-Penrose generalized inver $A^\dagger$ of A is introduced. It can be defined as the unique matrix satisfying the follow four matrix equations for X [2]:

$$AXA=A, \quad XAX=X, \quad (AX)^{\mathrm{T}}=AX, \quad (XA)^{\mathrm{T}}=XA.$$

The condition number for the generalized matrix inversion is given by $\|A\|\,\|A^\dagger\|$ [6]. For the problem of linear least squares

$$\min_x \|b-Ax\|, \tag{1.1}$$

the minimal norm solution is $A^\dagger b$ and the condition number is approximately $\|A\|\,\|A^\dagger\|$ when the residual $r=b-Ax$ is small and $\|A\|^2\|A^\dagger\|^2$ otherwise [6]. The condition numbers for weighted Moore-Penrose inverse and weighted least squares are discussed in [8, 9]. The condition numbers for structured least squares are given in [10].

The above condition numbers are called normwise condition numbers, because they are in the forms of matrix norms. The normwise analysis has two major drawbacks: It is norm dependent; it gives no information about the sensitivity of individual components [7]. Rohn [7] presented componentwise condition numbers for matrix inversion and nonsingular system of linear equations. Let $A=[A_{ij}]$. Denoting $|A|=[|A_{ij}|]$, we say $|A| \leq |B|$ when $|A_{ij}| \leq |B_{ij}|$ for all i and j. The componentwise condition number for matrix inversion is defined by

$$c_{ij}(A)=\lim_{\epsilon\to 0+}\sup\left\{\frac{|(A+\Delta A)^{-1}-A^{-1}|_{ij}}{\epsilon|A^{-1}|_{ij}},\ |\Delta A| \leq \epsilon|A|\right\},$$

for nonsingular $A+\Delta A$. Rohn proposed

$$c_{ij}(A)=\frac{(|A^{-1}|\,|A|\,|A^{-1}|)_{ij}}{|A^{-1}|_{ij}}. \tag{1.2}$$

For the nonsingular system $Ax=b$ of linear equations, Rohn defined

$$c_i(A,b)=\lim_{\epsilon\to 0+}\sup\left\{\frac{|(A+\Delta A)^{-1}(b+\Delta b)-A^{-1}b|_i}{\epsilon|A^{-1}b|_i},\ |\Delta A| \leq \epsilon|A|,\ |\Delta b| \leq \epsilon|b|\right\},$$