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A DIRECT ALGORITHM FOR DISTINGUISHING NONSINGULAR M-MATRIX AND H-MATRIX*

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Abstract *A direct algorithm is proposed by which one can distinguish whether a matrix is an M-matrix (or H-matrix) or not quickly and effectively. Numerical examples show that it is effective and convincible to distinguish M-matrix (or H-matrix) by using the algorithm.*

Key words *nonsingular M-matrix, nonsingular H-matrix, direct algorithm.*

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1 Introduction

For many kinds of applications of M-matrices and H-matrices, the problem how to determine whether a matrix is an M-matrix (or H-matrix) or not arouses many researchers interesting. Recently, some iterative methods have been proposed for distinguishing H-matrices (see [2-5]). However, these methods have a common drawback, that is, it is not possible to determine the number of steps of iteration, and when A is not an H-matrix, a wasteful computation is necessary. A direct algorithm has been proposed in [6], but it is only useful when matrices are symmetrical. In this paper, to conquer these drawbacks, we propose a new direct algorithm.

2 A direct algorithm for distinguishing M-matrix

Let $R^{n\times n}$ denote the set of all $n\times n$ real matrices. $A=(a_{ij})\in R^{n\times n}$ is said to be an M-matrix if $a_{ij}\le 0$, for $i\ne j$, and $A^{-1}\ge 0$.

Lemma 1[1] Let $A=(a_{ij})\in R^{n\times n}$ be an M-matrix, then any principle submatrix of A is an M-matrix.

Lemma 2[1] Let $A=(a_{ij})\in R^{n\times n}$, its off-diagonal entries are all non-positive, then A is

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an M-matrix if and only if successive principle minor of $A, D_K > 0, k = 1, \cdots, n$.

From Lemma 2, we can immediately obtain the following lemma.

Lemma 3 Let $A = (a_{ij}) \in R^{2\times 2}$, and $a_{ij} \leq 0, i \neq j, a_{ii} > 0$, then A is an M-matrix if and only if determinant of A, det $A > 0$.

Theorem 1 Let

$$B = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right] \in R^{n\times n},$$

where $B_{12} \leq 0, B_{21} \leq 0, B_{11}$ is a 2×2 square matrix and B_{22} is an $(n-2)\times(n-2)$ square matrix, in which their diagonal entries are all positive and off-diagonal entries are all non-positive. Then B is an M-matrix if and only if $\det B_{11} > 0$ and $B_{22} - B_{21}B_{11}^{-1}B_{12}$ is an M-matrix.

Proof Necessity: Suppose B is an M-matrix, then

$$B^{-1} = \left[\begin{array}{cc} B_{11}^{-1} + B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & -B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \\ -(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & (B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \end{array} \right] \geq 0,$$

and B_{11} and B_{22} are M-matrices by Lemma 1. Hence, $\det B_{11} > 0$ by Lemma 3, and $B_{11}^{-1} \geq 0, (B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \geq 0$. For $B_{12} \leq 0, B_{21} \leq 0$, we have $B_{21}B_{11}^{-1}B_{12} \geq 0$, and off-diagonal entries of matrix $B_{22} - B_{21}B_{11}^{-1}B_{12}$ are all non-positive. So, $B_{22} - B_2 B_{11}^{-1}B_{12}$ is an M-matrix.

Sufficiency: Suppose $\det B_{11} > 0$ and $B_{22} - B_{21}B_{11}^{-1}B_{12}$ is an M-matrix, then by Lemma 3, we have that B_{11} is an M-matrix, so

$$(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \geq 0, \qquad B_{11}^{-1} \geq 0.$$

Therefore

$$B_{11}^{-1} + B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} \geq 0,$$
$$-(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} \geq 0,$$
$$-B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \geq 0.$$

From these inequalities, we have

$$B^{-1} = \left[\begin{array}{cc} B_{11}^{-1} + B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & -B_{11}^{-1}B_{12}(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \\ -(B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1}B_{21}B_{11}^{-1} & (B_{22} - B_{21}B_{11}^{-1}B_{12})^{-1} \end{array} \right] \geq 0.$$

Thus B is an M-matrix.

From Theorem 1, we propose the following algorithm A.

Algorithm A

Input The given matrix $B = (b_{ij}) \in R^{n\times n}$.

Step 1 Set $B = B^{(m)}$, and $m = 0$.

Step 2 Partition $B^{(m)}$ into a 2×2 block matrix

$$B^{(m)} = \left[\begin{array}{cc} B_{11}^{(m)} & B_{12}^{(m)} \\ B_{21}^{(m)} & B_{22}^{(m)} \end{array} \right],$$