JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **32**, No. 1, pp. 52-65

doi: 10.4208/jpde.v32.n1.4
March 2019

Entire Large Solutions to Semilinear Elliptic Systems of Competitive Type

LAIR Alan V.*

Department of Mathematics and Statistics, Air Force Institute of Technology, Wright Patterson AFB, OH 45433-7765.

Received 18 April, 2018; Accepted 3 November, 2018

Abstract. We consider the elliptic system $\Delta u = p(|x|)u^a v^b$, $\Delta v = q(|x|)u^c v^d$ on $\mathbf{R}^n$ $(n \geq 3)$ where a, b, c, d are nonnegative constants with $\max\{a,d\} \leq 1$, and the functions p and q are nonnegative, continuous, and the support of $\min\{p(r),q(r)\}$ is not compact. We establish conditions on p and q, along with the exponents a, b, c, d, which ensure the existence of a positive entire solution satisfying $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = \infty$.

AMS Subject Classifications: 35J47, 35B44

Chinese Library Classifications: O175.25

Key Words: Large solution; entire solution; semilinear; elliptic system.

1 Introduction and main results

In this paper we establish the existence of positive solutions (u,v) to the elliptic system

$$\begin{aligned} &\Delta u = p(|x|)u^a v^b, \\ &\Delta v = q(|x|)u^c v^d, \quad x \in \mathbf{R}^n, \quad (n \geq 3), \end{aligned} \tag{1.1}$$

that satisfy

$$u(x) \to \infty \quad \text{and} \quad v(x) \to \infty \quad \text{as} \quad |x| \to \infty. \tag{1.2}$$

Such solutions of (1.1) are called entire large solutions. The exponents a,b,c d are nonnegative; the functions p, q are radial (i.e., spherically symmetric), nonnegative, and continuous; and the function $m(r) \equiv \min\{p(r),q(r)\}$ has noncompact support.

*Corresponding author. *Email addresses:* `alan.lair@afit.edu, alanlair@gmail.com` (A. V. Lair)

Although the existence of large solutions to semilinear systems began with [1], the study of large solutions to the more general competitive systems such as (1.1) started with García-Melián and Rossi [2] where the authors considered the system on a bounded domain with $\min\{a,d\}>1$ and unit weights (i.e., $p=q=1$). For both the subcritical case (i.e., $(a-1)(d-1)>bc$) and the critical case (i.e., $(a-1)(d-1)=bc$) necessary and sufficient conditions were given for the existence of boundary blow-up (or large) solutions. In addition, they established existence for the subcritical case when the weights are nonconstant, nonradial, and possibly blow up on the boundary with a prescribed asymptotic behavior. García-Melián [3] extended existence of blow-up solutions to the case where the weights, if unbounded, have prescribed growth rates at the boundary. Mu et al [4] also considered the subcritical case and proved existence when the weights are allowed to vanish on the boundary. Large solutions of the quasilinear problem where the Laplacian in (1.1) is replaced with the $p-$Laplacian have also been studied. (See, e.g., [5, 6]).

All of these results apply only to bounded domains. Here we study the existence of large solutions on all of $\mathbf{R}^n$ ($n\geq 3$). Except for special cases (e.g., [7] and [1] where $a=d=0$), the only other results known to the author is his work with Mohammed [8] where (1.1) is studied with unit weights and exponents that are radial functions of x. When applied to the present case where the exponents are constant, we proved that with unit weights a postive entire large solution exists if and only if $\max\{a,d\}\leq 1$ and $(1-a)(1-d)\leq bc$ ([8] Corollary 4.6). One consequence of this is, of course, that (1.1), with unit weights, will not have an entire large solution if $\min\{a,d\}>1$.

Before stating our results, we note some related problems that remain unsolved. For a nontrivial system (i.e., $bc>0$) with nonconstant nonradial weights, there is no known existence theorem for entire large solutions, even in the case where $a=d=0$. Even with nonconstant radial weights, as considered here, it remains unknown as to whether an entire large solution exists when $\min\{a,d\}>1$, regardless of the case: subcritical, critical, or supercritical (i.e., $(a-1)(d-1)>bc$). In particular, what are the appropriate conditions on the radial weights to ensure that such a solution exists? As mentioned above, the weights must be nonconstant in (1.1) since, otherwise, it will have a solution only if $\max\{a,d\}\leq 1$.

In order to state our main results we define G and H as follows where $P(r)=\int_0^r sp(s)\mathrm{d}s$ and $Q(r)=\int_0^r sq(s)\mathrm{d}s$ and note some equivalences (See (10) and (11) in [9]).

$$G(r)\equiv\int_0^r t^{1-n}\int_0^t s^{n-1}p(s)\mathrm{d}s\mathrm{d}t=r^{2-n}\int_0^r t^{n-3}\int_0^t sp(s)\mathrm{d}s\mathrm{d}t=r^{2-n}\int_0^r t^{n-3}P(t)\mathrm{d}t,$$
$$H(r)\equiv\int_0^r t^{1-n}\int_0^t s^{n-1}q(s)\mathrm{d}s\mathrm{d}t=r^{2-n}\int_0^r t^{n-3}\int_0^t sq(s)\mathrm{d}s\mathrm{d}t=r^{2-n}\int_0^r t^{n-3}Q(t)\mathrm{d}t.$$

Notice also that (See (12) and (13) of [9]).

$$\lim_{r\to\infty}G(r)=\infty \quad \text{if and only if} \quad \lim_{r\to\infty}P(r)=\infty, \tag{1.3}$$
$$\lim_{r\to\infty}H(r)=\infty \quad \text{if and only if} \quad \lim_{r\to\infty}Q(r)=\infty. \tag{1.4}$$