

Entire Large Solutions to Semilinear Elliptic Systems of Competitive Type

LAIR Alan V.*

*Department of Mathematics and Statistics, Air Force Institute of Technology,
Wright Patterson AFB, OH 45433-7765.*

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Abstract. We consider the elliptic system $\Delta u = p(|x|)u^a v^b$, $\Delta v = q(|x|)u^c v^d$ on \mathbf{R}^n ($n \geq 3$) where a, b, c, d are nonnegative constants with $\max\{a, d\} \leq 1$, and the functions p and q are nonnegative, continuous, and the support of $\min\{p(r), q(r)\}$ is not compact. We establish conditions on p and q , along with the exponents a, b, c, d , which ensure the existence of a positive entire solution satisfying $\lim_{|x| \rightarrow \infty} u(x) = \lim_{|x| \rightarrow \infty} v(x) = \infty$.

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1 Introduction and main results

In this paper we establish the existence of positive solutions (u, v) to the elliptic system

$$\begin{aligned} \Delta u &= p(|x|)u^a v^b, \\ \Delta v &= q(|x|)u^c v^d, \quad x \in \mathbf{R}^n, \quad (n \geq 3), \end{aligned} \tag{1.1}$$

that satisfy

$$u(x) \rightarrow \infty \quad \text{and} \quad v(x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty. \tag{1.2}$$

Such solutions of (1.1) are called entire large solutions. The exponents a, b, c, d are nonnegative; the functions p, q are radial (i.e., spherically symmetric), nonnegative, and continuous; and the function $m(r) \equiv \min\{p(r), q(r)\}$ has noncompact support.

*Corresponding author. *Email addresses:* alan.lair@afit.edu, alanlair@gmail.com (A. V. Lair)

Although the existence of large solutions to semilinear systems began with [1], the study of large solutions to the more general competitive systems such as (1.1) started with García-Melián and Rossi [2] where the authors considered the system on a bounded domain with $\min\{a, d\} > 1$ and unit weights (i.e., $p = q = 1$). For both the subcritical case (i.e., $(a-1)(d-1) > bc$) and the critical case (i.e., $(a-1)(d-1) = bc$) necessary and sufficient conditions were given for the existence of boundary blow-up (or large) solutions. In addition, they established existence for the subcritical case when the weights are non-constant, nonradial, and possibly blow up on the boundary with a prescribed asymptotic behavior. García-Melián [3] extended existence of blow-up solutions to the case where the weights, if unbounded, have prescribed growth rates at the boundary. Mu et al [4] also considered the subcritical case and proved existence when the weights are allowed to vanish on the boundary. Large solutions of the quasilinear problem where the Laplacian in (1.1) is replaced with the p -Laplacian have also been studied. (See, e.g., [5, 6]).

All of these results apply only to bounded domains. Here we study the existence of large solutions on all of \mathbf{R}^n ($n \geq 3$). Except for special cases (e.g., [7] and [1] where $a = d = 0$), the only other results known to the author is his work with Mohammed [8] where (1.1) is studied with unit weights and exponents that are radial functions of x . When applied to the present case where the exponents are constant, we proved that with unit weights a positive entire large solution exists if and only if $\max\{a, d\} \leq 1$ and $(1-a)(1-d) \leq bc$ ([8] Corollary 4.6). One consequence of this is, of course, that (1.1), with unit weights, will not have an entire large solution if $\min\{a, d\} > 1$.

Before stating our results, we note some related problems that remain unsolved. For a nontrivial system (i.e., $bc > 0$) with nonconstant nonradial weights, there is no known existence theorem for entire large solutions, even in the case where $a = d = 0$. Even with nonconstant radial weights, as considered here, it remains unknown as to whether an entire large solution exists when $\min\{a, d\} > 1$, regardless of the case: subcritical, critical, or supercritical (i.e., $(a-1)(d-1) > bc$). In particular, what are the appropriate conditions on the radial weights to ensure that such a solution exists? As mentioned above, the weights must be nonconstant in (1.1) since, otherwise, it will have a solution only if $\max\{a, d\} \leq 1$.

In order to state our main results we define G and H as follows where $P(r) = \int_0^r sp(s)ds$ and $Q(r) = \int_0^r sq(s)ds$ and note some equivalences (See (10) and (11) in [9]).

$$G(r) \equiv \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) ds dt = r^{2-n} \int_0^r t^{n-3} \int_0^t sp(s) ds dt = r^{2-n} \int_0^r t^{n-3} P(t) dt,$$

$$H(r) \equiv \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) ds dt = r^{2-n} \int_0^r t^{n-3} \int_0^t sq(s) ds dt = r^{2-n} \int_0^r t^{n-3} Q(t) dt.$$

Notice also that (See (12) and (13) of [9]).

$$\lim_{r \rightarrow \infty} G(r) = \infty \quad \text{if and only if} \quad \lim_{r \rightarrow \infty} P(r) = \infty, \quad (1.3)$$

$$\lim_{r \rightarrow \infty} H(r) = \infty \quad \text{if and only if} \quad \lim_{r \rightarrow \infty} Q(r) = \infty. \quad (1.4)$$

Our first result extends Theorem 4.3 of [8] where $p = q = 1$, and both of the results below extend Theorem 2 of [9] where $a = d = 0$.

Theorem 1.1. Assume $\max\{a, d\} \leq 1$ and $bc \leq (1-a)(1-d)$. If p and q satisfy both

$$\int_0^\infty rp(r)H^b(r)dr = \infty, \quad (1.5)$$

$$\int_0^\infty rq(r)G^c(r)dr = \infty, \quad (1.6)$$

then (1.1) has a positive entire large solution.

It remains open as to whether the conditions (1.5) and (1.6) are necessary; i.e., if either (1.5) or (1.6) fails to hold, then it is unknown as to whether (1.1) has a positive entire large solution.

Theorem 1.2. Assume $\max\{a, d\} \leq 1$ and $bc > (1-a)(1-d)$. If $\max\{a, d\} < 1$ and if p and q satisfy either

$$\int_0^\infty rp(r)H^\alpha(r)dr < \infty, \quad (1.7)$$

or

$$\int_0^\infty rq(r)G^\beta(r)dr < \infty, \quad (1.8)$$

where $\alpha = b/(1-d)$ and $\beta = c/(1-a)$, then (1.1) has a positive entire large solution. On the other hand, if $\max\{a, d\} = 1$ (and hence either (1.7) or (1.8) is undefined) and if p and q satisfy

$$\int_0^\infty rp(r)dt < \infty \quad \text{and} \quad \int_0^\infty rq(r)dt < \infty, \quad (1.9)$$

then (1.1) has a positive entire large solution.

2 Auxiliary results and proofs

Before proving our results, we note first that for $u_0 > 0$, $v_0 > 0$, any solution of the system

$$u(r) = u_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u^a(s) v^b(s) ds dt, \quad (2.1)$$

$$v(r) = v_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u^c(s) v^d(s) ds dt \quad (2.2)$$

valid for all $r \geq 0$ will also be a positive entire solution to (1.1) but not necessarily (1.2).

Proof of Theorem 1.1. We first consider the case $\max\{a, d\} = 1$, and we assume with no loss of generality that $a = 1$ and $d \leq 1$. Since, by hypothesis, $bc \leq (1-a)(1-d)$, we get $bc = 0$. We assume $b = 0$, and note that if instead $c = 0$ the proof is similar. Thus (1.1) becomes

$$\Delta u = p(|x|)u, \quad \Delta v = q(|x|)u^c v^d. \quad (2.3)$$

The first of these equations has a positive entire large radial solution u since p satisfies (1.5) with $b=0$ (See Theorem 1 of [10]), and it satisfies, for any $u_0 > 0$,

$$u(r) = u_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u(s) ds dt.$$

Since u is positive, the right side of this equation gives $u \geq u_0$, and substituting this back into the equation yields

$$u(r) \geq u_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_0 ds dt \geq u_0 G(r). \quad (2.4)$$

Now with this solution u used in the second equation in (2.3), we define the sequence $\{v_k\}$ as follows: let v_0 to be any constant $v_0 \geq 1$ and

$$v_k(r) = v_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u^c(s) v_{k-1}^d(s) ds dt \quad (k \geq 1). \quad (2.5)$$

Since $1 \leq v_0 < v_1$ and hence $1 \leq v_1 < v_2$, it is clear that the sequence $\{v_k\}$ is increasing. Furthermore, using integration by parts and the monotonicity of $\{v_k\}$, we get

$$\begin{aligned} v_k(r) &= v_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u^c(s) v_{k-1}^d(s) ds dt \\ &= v_0 + \frac{1}{n-2} \int_0^r s q(s) u^c(s) v_{k-1}^d(s) ds - \frac{1}{n-2} r^{2-n} \int_0^t s^{n-1} q(s) u^c(s) v_{k-1}^d(s) ds dt \\ &\leq v_0 + \int_0^r s q(s) u^c(s) v_{k-1}^d(s) ds \leq v_0 + \int_0^r s q(s) u^c(s) v_k(s) ds \end{aligned}$$

where the last inequality follows from the monotonicity of $\{v_k\}$ so that $v_{k-1}^d \leq v_k^d$ and $v_k^d \leq v_k$ since $v_k \geq 1$ and $d \leq 1$. Now Gronwall's inequality provides an upper bound for the sequence $\{v_k\}$ in terms of u . Hence $\{v_k\}$ converges for all $r \geq 0$; let $\lim_{k \rightarrow \infty} v_k = v$. Clearly, v satisfies (2.2), and hence (u, v) satisfies (2.3). We now show that v is large. Since $b=0$, equation (1.5) reduces to $P(\infty) = \infty$ which, by (1.3), is equivalent to $G(\infty) = \infty$. Therefore, using (1.6), (2.4), and elementary estimates we get

$$\begin{aligned} v(r) &= v_0 + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u^c(s) v^d(s) ds dt \\ &\geq u_0^c v_0^d \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) G^c(s) ds dt \\ &\geq \frac{u_0^c v_0^d}{2(n-2)} \int_0^{r/2} t q(t) G^c(t) dt \rightarrow \infty \quad \text{as } r \rightarrow \infty. \end{aligned} \quad (2.6)$$

The last inequality can be easily obtained using integration by parts (see p. 748 of [10]).

Now suppose $\max\{a, d\} < 1$. By hypothesis then $\alpha\beta = \frac{b}{1-d} \frac{c}{1-a} \leq 1$, and consequently, either $\alpha \leq 1$ or $\beta \leq 1$. We assume without loss of generality that $\beta \leq 1$. We first show that

system (1.1) has an entire solution and then show that it is large. Define sequences $\{u_k\}$ and $\{v_k\}$ as follows with $u_0 = v_0 = \rho$ and for any $\rho \geq 1$

$$u_k(r) = \rho + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_{k-1}^a(s) v_{k-1}^b(s) ds dt, \quad (2.7)$$

$$v_k(r) = \rho + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u_k^c(s) v_{k-1}^d(s) ds dt. \quad (2.8)$$

Any solution to this system will also be a solution to the system

$$\Delta u_k = p(|x|) u_{k-1}^a v_{k-1}^b \quad \Delta v_k = q(|x|) u_k^c v_{k-1}^d. \quad (2.9)$$

Note that both sequences are monotonically increasing since obviously $u_0 < u_1$ and $v_0 < v_1$ which, in turn, yields $u_1 < u_2$ so that $v_1 < v_2$. Continuing in this manner produces the monotonicity of the two sequences. We will show that both sequences are bounded above on an arbitrary bounded interval $[0, R]$ and hence the limit will be a solution to (2.1) and (2.2). To do this, let $w_k = u_k^{1-a}$ and $z_k = v_k^{1-d}$. It is easy to see that ($r \equiv |x|$)

$$\begin{aligned} \Delta w_k &= -a(1-a) u_k^{-1-a} |\nabla u_k|^2 + (1-a) u_k^{-a} \Delta u_k \leq (1-a) u_k^{-a} \Delta u_k \\ &= (1-a) p(r) u_k^{-a} u_{k-1}^a v_{k-1}^b \leq (1-a) p(r) v_{k-1}^b \leq p(r) z_{k-1}^\alpha. \end{aligned} \quad (2.10)$$

Similarly, we can get

$$\Delta z_k \leq q(r) w_k^\beta. \quad (2.11)$$

Integrating these inequalities we obtain

$$w_k(r) \leq \rho^{1-a} + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) z_{k-1}^\alpha(s) ds dt, \quad (2.12)$$

$$z_k(r) \leq \rho^{1-d} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) w_k^\beta(s) ds dt. \quad (2.13)$$

Clearly the sequences $\{w_k\}$ and $\{z_k\}$, like $\{u_k\}$ and $\{v_k\}$, are monotonically increasing so if we can show that both are bounded above on $[0, R]$, then both converge on $[0, \infty)$. Consequently, $\{u_k\}$ and $\{v_k\}$ would converge. From the monotonicity of the sequence $\{z_k\}$ of monotonically increasing functions, inequality (2.12) yields

$$w_k(r) \leq \rho^{1-a} + \left(\int_0^r t^{1-n} \int_0^t s^{n-1} p(s) ds dt \right) z_k^\alpha(r) = \rho^{1-a} + G(r) z_k^\alpha(r). \quad (2.14)$$

We substitute this into (2.13) and use the well known inequality $(A+B)^\beta \leq A^\beta + B^\beta$ which holds for any nonnegative A and B since $\beta \leq 1$ to get

$$z_k(r) \leq \rho^{1-d} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) \left(\rho^{1-a} + G(s) z_k^\alpha(s) \right)^\beta ds dt$$

$$\begin{aligned} &\leq \rho^{1-d} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) \left(\rho^{\beta(1-a)} + G^\beta(s) z_k^{\alpha\beta}(s) \right) ds dt \\ &= \rho^{1-d} + \rho^{\beta(1-a)} H(r) + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) G^\beta(s) z_k^{\alpha\beta}(s) ds dt. \end{aligned}$$

Recalling that $\alpha\beta \leq 1$ and that $z_k \geq 1$ since $\rho \geq 1$, we get

$$\begin{aligned} z_k(r) &\leq \rho^{1-d} + \rho^{\beta(1-a)} H(r) + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) G^\beta(s) z_k(s) ds dt \\ &\leq C_R + \int_0^r s q(s) G^\beta(s) z_k(s) ds dt \end{aligned}$$

where $C_R = \rho^{1-d} + \rho^{\beta(1-a)} H(R)$. Gronwall's inequality may now be used to get $\{z_k\}$ bounded independently of k on $[0, R]$, and hence on any bounded interval. Consequently, (2.14) gives $\{w_k\}$ bounded also on any bounded interval. Therefore we obtain the existence of a positive entire solution to (2.1) and (2.2) (and (1.1)) for any $\rho \geq 1$.

To show that these solutions are large, we note that as a consequence of (1.5) and (1.6), we must have either $P(\infty) = \infty$ or $Q(\infty) = \infty$. We assume with no loss in generality that $P(\infty) = \infty$ and thus $G(\infty) = \infty$ so that using $u \geq \rho$ and $v \geq \rho$ in (2.1) (with $u_0 = v_0 = \rho$) yields

$$u(r) \geq \rho + \rho^{a+b} G(r) \rightarrow \infty \quad \text{as } r \rightarrow \infty.$$

For v we note that from (2.1), we get

$$u(r) \geq \rho^a \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) v^b(s) ds dt$$

which, when substituted into (2.2) and using elementary estimates, yields

$$\begin{aligned} v(r) &\geq \rho + \rho^{ac} \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) \left(\int_0^s \xi^{1-n} \int_0^\xi \tau^{n-1} p(\tau) v^b(\tau) d\tau d\xi \right)^c v^d(s) ds dt \\ &\geq \rho + \rho^{ac+bc+d} \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) \left(\int_0^s \xi^{1-n} \int_0^\xi \tau^{n-1} p(\tau) d\tau d\xi \right)^c ds dt \\ &= \rho + \rho^{ac+bc+d} \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) G^c(s) ds dt. \end{aligned}$$

Applying integration by parts as in (2.6) and using simple estimates along with (1.6) we get

$$v(r) \geq \rho + \frac{\rho^{ac+bc+d}}{2(n-2)} \int_0^{r/2} t q(t) G^c(t) dt \rightarrow \infty \quad \text{as } r \rightarrow \infty.$$

This completes the proof. \square

Before proving Theorem 1.2, we need to establish a preliminary result on the existence of a large solution of (1.1) on a ball of finite radius. Our proof will somewhat parallel those of [9] (see Theorem 3.1) and [8]. There are, however, important differences since $a = d = 0$ in [9] and $p = q = 1$ in [8].

Lemma 2.1. Assume $\max\{a, d\} < 1$, α and β are defined as in Theorem 1.2, and let R be any positive number for which $m(R) > 0$. If $bc > (1-a)(1-d)$, then the system (1.1) has a positive large solution on the ball $|x| \leq R$.

Proof. We show that the system (2.1) and (2.2) has a large solution on $|x| \leq R$ provided $\rho > 0$ is chosen appropriately. First we show that the system has a solution valid on $|x| \leq R$ and then establish that ρ can be chosen so that that solution is large. To do this let ζ be a positive large solution of

$$\Delta\zeta = h(r)\zeta^{\alpha\beta}$$

on $|x| \leq R$ where $h(r) = 2^\beta q(r)P^\beta(r)$. Such a solution exists (See Theorem 1 of [11]) since $h(R) > 0$, and $bc > (1-a)(1-d)$ so that $\alpha\beta > 1$. Now choose $\rho > 0$ small so that

$$\rho^{1-d} + 2^\beta \rho^{\beta(1-a)} H(R) < \zeta(0). \quad (2.15)$$

With this value of ρ , we define sequences $\{u_k\}$ and $\{v_k\}$ as in (2.7) and (2.8) for $r \in [0, R]$. As noted earlier these sequences are monotonically increasing so if we can show that they are bounded independently of k , then they must converge to a solution (u, v) , not necessarily large, on the same ball. Letting $w_k = u_k^{1-a}$, $z_k = v_k^{1-d}$ as in (2.10) and (2.11), and using (2.12) along with the monotonicity of $\{z_k\}$ and integration by parts, we get

$$\begin{aligned} w_k(r) &\leq \rho^{1-a} + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) z_{k-1}^\alpha(s) ds dt \\ &\leq \rho^{1-a} + \int_0^r t p(t) z_{k-1}^\alpha(t) dt \leq \rho^{1-a} + \int_0^r t p(t) z_k^\alpha(t) dt \\ &\leq \rho^{1-a} + P(r) z_k^\alpha(r). \end{aligned} \quad (2.16)$$

Using this inequality in (2.13), we get

$$\begin{aligned} z_k(r) &\leq \rho^{1-d} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) \left(\rho^{1-a} + P(s) z_k^\alpha(s) \right)^\beta ds dt \\ &\leq \rho^{1-d} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) 2^\beta \left(\rho^{\beta(1-a)} + P^\beta(s) z_k^{\alpha\beta}(s) \right) ds dt \\ &\leq \rho^{1-d} + 2^\beta \rho^{\beta(1-a)} H(R) + 2^\beta \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) P^\beta(s) z_k^{\alpha\beta}(s) ds dt. \end{aligned}$$

By our choice of ρ in (2.15), it is clear that $z_k(r) < \zeta(r)$ for r small. Letting $R_0 = \sup\{r_0 > 0 \mid z_k(r) < \zeta(r) \text{ for all } r \in [0, r_0]\}$, we can use analysis very similar to that in [9] (see p. 327) to prove that $R_0 = R$ so that $z_k < \zeta$ on $[0, R]$, and hence $\{z_k\}$ (and consequently $\{v_k\}$) converges on $[0, R]$. Using (2.16) we also get w_k (and consequently $\{u_k\}$) convergent on $[0, R]$. Hence $\{u_k\}$ and $\{v_k\}$ converge on $[0, R]$ and therefore the system (2.1) and (2.2) has a positive entire solution (u, v) for ρ small.

We now show that ρ can be chosen so that the solution obtained is large. Since the details of the proof are similar to those of [9] (see proof of the lemma) and [8] (see proof

of Theorem 3.1), we merely outline the proof here. Define the set T by

$$T = \{\rho > 0 \mid (2.1), (2.2) \text{ has a solution on } [0, R]\},$$

and we show that T is bounded. To do this, define the constant γ by (see (3.3) in [8])

$$\gamma = a + \frac{(1+b-a)(1+c-a)}{2+b+c-a-d} > 1,$$

and let R_1 be any number in the open interval $(0, R)$ for which $m(R_1) > 0$. Then let ξ be a positive large solution of $\Delta \xi = m(|x|)\xi^\gamma$ on $|x| < R_1$. It then follows (see p. 327 of [9]) that $\sqrt{\max\{1, \xi(0)\}}$ is an upper bound for T . Likewise it also follows (see [8] p. 1486) that $A \equiv \sup(T) \in T$, and the solution of (2.1) and (2.2) corresponding to $\rho = A$ is large. This completes the proof. \square

Proof of Theorem 1.2. We first consider the case $\max\{a, d\} < 1$ and will assume that (1.7) holds; the proof when (1.8) holds is similar and therefore omitted. Let r_k be any increasing sequence of positive numbers diverging to infinity for which $m(r_k) > 0$ for each k . Let (u_k, v_k) be a positive large solution to (1.1) on the ball $|x| < r_k$ which was established by the lemma and satisfies

$$u_k(r) = \rho_k + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_k^a(s) v_k^b(s) ds dt, \quad (2.17)$$

$$v_k(r) = \rho_k + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u_k^c(s) v_k^d(s) ds dt. \quad (2.18)$$

We show first that this sequence is monotonically decreasing on $[0, r_k]$; i.e.,

$$u_{k+1}(r) < u_k(r) \quad , \quad v_{k+1}(r) < v_k(r) \quad \text{for all } r \in [0, r_k]. \quad (2.19)$$

(For brevity, we write $(u_{k+1}, v_{k+1}) < (u_k, v_k)$). In particular, we show that $(u_2, v_2) < (u_1, v_1)$ on $[0, r_1]$; a similar proof, which we omit, gives $(u_{k+1}, v_{k+1}) < (u_k, v_k)$ on $[0, r_k]$. Obviously $\rho_1 \neq \rho_2$, otherwise $(u_2, v_2) = (u_1, v_1)$ on $[0, r_1]$, which is impossible since (u_1, v_1) blows up at r_1 and (u_2, v_2) does not. Thus suppose $\rho_1 < \rho_2$, and let $R = \sup(S)$ where $S \equiv \{\eta \in [0, r_1] : (u_1(r), v_1(r)) < (u_2(r), v_2(r)) \text{ for all } r \in [0, \eta]\}$. The set S is clearly nonempty since $0 \in S$ and thus $0 < R \leq r_1$. If $R = r_1$, then we have a contradiction since that would mean that $\lim_{r \rightarrow r_1} u_2(r) = \lim_{r \rightarrow r_1} v_2(r) = \infty$ which cannot occur since (u_2, v_2) is continuous on $[0, r_1] \subseteq [0, r_2]$. So, suppose $R < r_1$. Then

$$\begin{aligned} u_1(R) &= \rho_1 + \int_0^R t^{1-n} \int_0^t s^{n-1} p(s) u_1^a(s) v_1^b(s) ds dt \\ &\leq \rho_1 + \int_0^R t^{1-n} \int_0^t s^{n-1} p(s) u_2^a(s) v_2^b(s) ds dt \\ &< \rho_2 + \int_0^R t^{1-n} \int_0^t s^{n-1} p(s) u_2^a(s) v_2^b(s) ds dt = u_2(R). \end{aligned}$$

Thus $u_1 < u_2$ on $[0, R]$. Similarly, we can get $v_1 < v_2$ on $[0, R]$. Thus there exists $\varepsilon > 0$ so that $(u_1, v_1) < (u_2, v_2)$ on $[0, R + \varepsilon]$ which contradicts the definition of R . Hence we must have $(u_2, v_2) < (u_1, v_1)$ on $[0, r_1]$. A similar proof produces $(u_{k+1}, v_{k+1}) < (u_k, v_k)$ on $[0, r_k]$ for all $k \in \mathbf{N}$.

It is thus clear that the positive decreasing sequence $\{(u_k, v_k)\}$ has a limit (u, v) on \mathbf{R}^n and that (u, v) satisfies (2.1), (2.2). We need to show that (u, v) is both positive and large. To this end, let Θ be a positive entire large solution of $(v \equiv (1 - d) / (1 - a))$

$$\Delta \Theta = p(r)(1 + H(r))^\alpha (\Theta^\nu + \Theta^\beta)^\alpha.$$

Such a solution exists (see Theorem 2 of [7]) since $\alpha\beta > 1$ and

$$\int_0^\infty rp(r)(1 + H(r))^\alpha dr < \infty.$$

As in the proof of Theorem 1.1, we define $w_k = u_k^{1-a}$ and $z_k = v_k^{1-d}$ so that, similar to (2.10) and (2.11), we get

$$\Delta w_k \leq p(r)z_k^\alpha, \quad \Delta z_k \leq q(r)w_k^\beta, \tag{2.20}$$

which yields

$$w_k(r) \leq \rho_k^{1-a} + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s)z_k^\alpha(s) ds dt, \tag{2.21}$$

$$z_k(r) \leq \rho_k^{1-d} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s)w_k^\beta(s) ds dt. \tag{2.22}$$

From (2.22) and the fact that w_k is an increasing function, we get

$$z_k(r) \leq \rho_k^{1-d} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s)w_k^\beta(s) ds dt \leq w_k^\nu(r) + H(r)w_k^\beta(r).$$

Using this in (2.20), we get

$$\Delta w_k \leq p(r) \left(w_k^\nu + H(r)w_k^\beta \right)^\alpha \leq p(r)(1 + H(r))^\alpha (w_k + w_k^\beta)^\alpha \quad \text{on } [0, r_k].$$

This inequality and the maximum principle along with the property $w_k(r) \rightarrow \infty$ as $r \rightarrow r_k$ shows that $\Theta \leq w_k$ on $[0, r_k)$ which yields $\Theta \leq u^{1-a}$ on \mathbf{R}^n , and hence u is positive and $u(r) \rightarrow \infty$ as $r \rightarrow \infty$.

To show that v is also both positive and large, we first note that since $u_k(0) = v_k(0)$ for all k , and $u(0) > 0$, then clearly $v(0) > 0$ so v is positive. To prove v is large, we consider two cases: $Q(\infty) < \infty$ and $Q(\infty) = \infty$. If $Q(\infty) < \infty$, then $H(\infty) < \infty$ by (1.4). Consequently since (1.7) holds we must have $P(\infty) < \infty$, and thus (1.3) gives $G(\infty) < \infty$. Therefore (1.8)

holds and an argument very similar to the proof above that u is large will give v large. If, on the other hand, $Q(\infty) = \infty$, then (1.4) gives $H(\infty) = \infty$ so that (2.2) yields

$$v(r) > \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u^c(s) v^d(s) ds dt \geq u^c(0) v^d(0) H(r) \rightarrow \infty \text{ as } r \rightarrow \infty.$$

For the case $\max\{a, d\} = 1$, we assume for simplicity that $a = 1$ and $d \leq 1$ and use the sequences $\{u_k\}$ and $\{v_k\}$ from (2.7) and (2.8) except that the constant ρ is merely positive. Clearly the sequences are monotonically increasing, but we must show that, for ρ small, they are bounded above for all $r \geq 0$ and therefore converge to an entire solution of (2.1), (2.2). Then we must prove that there is a value ρ for which the positive limit function (u, v) is both entire and large. Let w be an positive entire large solution of $\Delta w = (p(r) + q(r))(w^{1+b} + w^{c+d})$ (see Theorem 2 of [7]) and choose $\rho < w(0)$. Then $u_1 < w$ since $(u_0 = v_0 = \rho)$

$$\begin{aligned} u_1(r) &= \rho + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_0(s) v_0^b(s) ds dt \\ &\leq \rho + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) w^{1+b}(s) ds dt \\ &< w(0) + \int_0^r t^{1-n} \int_0^t s^{n-1} (p(s) + q(s)) (w^{1+b}(s) + w^{c+d}(s)) ds dt = w(r). \end{aligned}$$

Similarly, we get $v_1 < w$ which, in turn, will yield $u_k \leq w$ and $v_k \leq w$ for all k . Thus the sequences $\{u_k\}$ and $\{v_k\}$ converge for ρ sufficiently small.

We now show that there exists $\rho_0 > 0$ for which the corresponding solution is both entire and large. To do this, we first note that it is easy to prove that for $u_0 = v_0 = \rho$ in (2.1) and (2.2), the solution (u_ρ, v_ρ) is monotonically increasing in ρ . To establish the existence of ρ_0 we first prove that there exists $\gamma > 0$ such that the solution (u_γ, v_γ) is not entire. Without loss of generality, we assume that $0 < c \leq d$ and let $\delta = bc/(1+b)$. Then there is a positive solution z to (see Theorem 1 of [11])

$$\Delta z = m(r) z^{1+\delta} \quad \text{on } [0, 1) \text{ with } z(r) \rightarrow \infty \text{ as } r \rightarrow 1^-. \quad (2.23)$$

Choose $\gamma^2 > \max\{1, z(0)\}$, and we show that (u_γ, v_γ) is not entire. It is obvious that $z(0) < \gamma^2 = u_\gamma(0)v_\gamma(0)$, and therefore there exists $\eta > 0$ for which $z(r) < u_\gamma(r)v_\gamma(r)$ on $[0, \eta)$. Let $R = \sup\{\eta \mid z(r) < u_\gamma(r)v_\gamma(r) \text{ for all } r \in [0, \eta)\}$. If $R = 1$, then we are done since that shows $u_\gamma(r)v_\gamma(r) \rightarrow \infty$ as $r \rightarrow 1$, and hence (u_γ, v_γ) cannot be entire. Thus assume $R < 1$. Then (we now drop the subscript γ for simplicity)

$$\begin{aligned} z(R) &= z(0) + \int_0^R t^{1-n} \int_0^t s^{n-1} m(s) z^{1+\delta}(s) ds dt \\ &\leq z(0) + \int_0^R t^{1-n} \int_0^t s^{n-1} m(s) (u(s)v(s))^{1+\delta} ds dt. \end{aligned} \quad (2.24)$$

However, since $\delta/c + 1/(1+b) = 1$, $0 < c \leq d$, and $v \geq 1$, we get

$$(uv)^{1+\delta} = (u^c v^c)^{\frac{\delta}{c}} (v^{1+b})^{\frac{1}{1+b}} u \leq \left(\frac{\delta}{c} (uv)^c + \frac{1}{1+b} v^{1+b} \right) u \leq u^{1+c} v^d + u v^{1+b}.$$

Using this in (2.24), we get

$$\begin{aligned} z(R) &\leq z(0) + \int_0^R t^{1-n} \int_0^t s^{n-1} m(s) \left(u^{1+c}(s) v^d(s) + u(s) v^{1+b}(s) \right) ds dt \\ &< \gamma^2 + \int_0^R t^{1-n} \int_0^t s^{n-1} m(s) \left(u^{1+c}(s) v^d(s) + u(s) v^{1+b}(s) \right) ds dt. \end{aligned} \quad (2.25)$$

However, $\Delta(uv) = u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u \geq u\Delta v + v\Delta u = pu v^{1+b} + qu^{1+c} v^d$ so that integration gives

$$\begin{aligned} u(r)v(r) &\geq u(0)v(0) + \int_0^r t^{1-n} \int_0^t s^{n-1} \left(p(s)u(s)v^{1+b}(s) + q(s)u^{1+c}(s)v^d(s) \right) ds dt \\ &\geq \gamma^2 + \int_0^r t^{1-n} \int_0^t s^{n-1} m(s) \left(u(s)v^{1+b}(s) + u^{1+c}(s)v^d(s) \right) ds dt. \end{aligned}$$

Using this in (2.25), we get $z < uv$ on $[0, R]$ so there must exist $\varepsilon > 0$ such that $z < uv$ on $[0, R + \varepsilon)$, contradicting the definition of R . Therefore, it must be that $R = 1$, and hence $u(r)v(r) \rightarrow \infty$ as $r \rightarrow 1$ so (u, v) is not entire for this choice of γ and hence for any larger value. Thus the set $T \equiv \{\rho \mid (u_\rho, v_\rho) \text{ is entire}\}$ is bounded; let $A = \sup T$. We show now that (u_A, v_A) , the solution when $\rho = A$, is entire and large.

To prove that (u, v) (We drop the subscript A) is entire, we show for an arbitrary $R > 0$ that (u, v) is finite on $[0, R)$. To do this we define the monotonically increasing sequence (u_k, v_k) of entire functions defined by

$$\begin{aligned} u_k(r) &= A - \frac{1}{k} + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_k(s) v_k^b(s) ds dt, \\ v_k(r) &= A - \frac{1}{k} + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_k^c(s) v_k^d(s) ds dt, \end{aligned}$$

and show that (u_k, v_k) is bounded above $[0, R]$, and therefore its limit (u, v) is bounded above. To do this let z satisfy the equation in (2.23) on $[0, R+1)$ with $z(R+1) = \infty$. From the work above it is clear that $\Delta(u_k v_k - z) \geq m(r) \left((u_k v_k)^{1+\delta} - z^{1+\delta} \right)$ for all $r < R+1$, and since $u_k v_k - z < 0$ near $r = R+1$, the maximum principle can be used to show that $u_k v_k \leq z$ on $[0, R+1)$. Hence uv and therefore (u, v) does not blow up in the interval $[0, R]$. Since R was chosen arbitrarily, we conclude that (u, v) is entire.

To show that (u, v) is large, we define the sequence (u_k, v_k) by

$$u_k(r) = A + \frac{1}{k} + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_k(s) v_k^b(s) ds dt, \quad (2.26)$$

$$v_k(r) = A + \frac{1}{k} + \int_0^r t^{1-n} \int_0^t s^{n-1} p(s) u_k^c(s) v_k^d(s) ds dt. \quad (2.27)$$

Since (u_k, v_k) is not entire, at least one of the functions blows up at some $R_k < \infty$; we assume $u_k(r) \rightarrow \infty$ as $r \rightarrow R_k$. However, it must also be that $v_k(r) \rightarrow \infty$ as $r \rightarrow R_k$ since (2.26) yields the estimate

$$u_k(r) \leq A + \frac{1}{k} + \int_0^r tp(t) u_k(t) v_k^b(t) dt$$

so that Gronwall's inequality applied to this produces

$$u_k(r) \leq A + \frac{1}{k} + \int_0^r tp(t) v_k^b(t) \exp \left[\int_t^r sp(s) v_k^b(s) ds \right] dt, \quad (2.28)$$

and therefore v_k blows up at R_k since u_k does. Furthermore, it is easy to show that $(u_{k+1}, v_{k+1}) < (u_k, v_k)$ and hence $R_k \leq R_{k+1}$. Using (2.27), we get

$$\begin{aligned} v_k(r) &= A + \frac{1}{k} + \int_0^r t^{1-n} \int_0^t s^{n-1} q(s) u_k^c(s) v_k^d(s) ds dt \\ &\leq A + \frac{1}{k} + \int_0^r tq(t) u_k^c(t) (1 + v_k(t)) dt = g_k(r) + \int_0^r tq(t) u_k^c(t) v_k(t) dt, \end{aligned}$$

where $g_k(r) = A + \frac{1}{k} + \int_0^r tq(t) u_k^c(t) dt$ so that Gronwall's inequality gives

$$v_k(r) \leq g_k(r) + \int_0^r tq(t) u_k^c(t) g_k(t) \exp \left[\int_t^r sq(s) u_k^c(s) ds \right] dt. \quad (2.29)$$

Using (1.9), we get the existence of a constant C_1 such that $g_k(r) \leq C_1 u_k^c(r)$. Substituting this into (2.29) and using (1.9) once again gives the existence of constants C and K , independent of k , such that

$$v_k(r) \leq C \exp(K u_k^c(r)).$$

Substituting this into the first equation of (1.1), we get

$$\Delta u_k \leq C^b p(r) u_k \exp(b K u_k^c(r)) \equiv p(r) f(u_k). \quad (2.30)$$

Define

$$F(s) = \int_s^\infty \frac{dt}{f(t)}$$

and note that $F'(s) = -1/f(s) < 0$ and $F'' > 0$. Using this along with (2.30) gives

$$\Delta F(u_k) = F'(u_k) \Delta u_k + F''(u_k) |\nabla u_k|^2 \geq F'(u_k) \Delta u_k \geq -p(r),$$

or equivalently

$$\frac{d}{dr} \left(r^{n-1} \frac{d}{dr} F(u_k(r)) \right) \geq -r^{n-1} p(r).$$

Integrating this over $[0, r]$ for $r < R_k$, we get

$$\frac{d}{dr}F(u_k(r)) \geq -r^{1-n} \int_0^r t^{n-1} p(t) dt.$$

Noting that $u_k(R_k) = \infty$ so that $F(u_k(R_k)) = 0$, we integrate over $[r, R_k]$ to get

$$F(u_k(r)) \leq \int_r^{R_k} s^{1-n} \int_0^s t^{n-1} p(t) dt ds.$$

However, since $F'(s) < 0$ we get

$$u_k(r) \geq F^{-1} \left(\int_r^{R_k} s^{1-n} \int_0^s t^{n-1} p(t) dt ds \right)$$

and letting $k \rightarrow \infty$ so that $R_k \rightarrow R \leq \infty$ and $u_k \rightarrow u$ produces

$$u(r) \geq F^{-1} \left(\int_r^R s^{1-n} \int_0^s t^{n-1} p(t) dt ds \right).$$

Letting $r \rightarrow R$ and using $F^{-1}(s) \rightarrow \infty$ as $s \rightarrow 0$, we have

$$\lim_{r \rightarrow R} u(r) \geq \lim_{r \rightarrow R} F^{-1} \left(\int_r^R s^{1-n} \int_0^s t^{n-1} p(t) dt ds \right) = \infty.$$

However, u is entire so we conclude that $R = \infty$ and $\lim_{r \rightarrow \infty} u(r) = \infty$. Using an argument similar to that which produced (2.28), it can be shown that v also blows up at infinity. This completes the proof.

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