Fractional Tikhonov Regularization Method for a Time-Fractional Backward Heat Equation with a Fractional Laplacian

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Abstract. In this paper, we consider a time-fractional backward problem with a fractional Laplacian. We propose a fractional Tikhonov regularization method for solving this problem under the a-priori parameter choice rule. Error estimates are proved. Some numerical examples are shown to verify the effectiveness of the proposed method.

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1 Introduction

The backward heat conduction problem (BHCP) has been studied by many researchers [1,2]. Also there are many mathematical results about fractional backward heat condition problem (FBHCP) in the domain $\mathcal{R}^d$. Deng and Yang [3] considered a discretized Tikhonov regularization method for solving it, Li and Xiong [4] gave a general regularization method for the FBHCP with application to a deblurring problem. However there are few mathematical results about a time-fractional backward heat equation with fractional Laplacian in a bounded domain. Recently some fractional regularization methods are developed for solving all kinds of ill-posed problems. Please refer to [5-11].

Let $\Omega$ be a bound domain in $\mathcal{R}^d$ ($d \in \mathcal{N}$) with sufficient smooth boundary $\partial \Omega$, now we consider the following backward heat equation with fractional Laplacian as follows:

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http://www.global-sci.org/jpde/
\begin{align}
\begin{cases}
D_\alpha^t u(x,t) = -A^\beta u(x,t), & (0,T) \times \Omega, \\
u(x,T) = g(x), & \Omega, \\
u(x,t) = 0, & (0,T) \times \partial \Omega,
\end{cases}
\end{align}
\tag{1.1}

where \(0 \leq \alpha \leq 1, 0 \leq \beta \leq 1\) and the Dirichlet-Laplacian \(A\) is
\begin{align}
\begin{cases}
Af := -\Delta f = -\sum_{j=1}^d \partial_j^2 f, \\
D(A) := L^2(\Omega) \cap H_0^1(\Omega) \cap W^{2,2},
\end{cases}
\end{align}
\tag{1.2}

here \(W^{m,p}(\Omega)\) is the Sobolev space.

And \(D_\alpha^t\) is the Caputo fractional derivative of order \(\alpha\) \((0 \leq \alpha \leq 1)\) defined by
\begin{align}
D_\alpha^t u(x,t) &= \begin{cases}
\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{\alpha-1} u_\tau(x,\tau) \, d\tau, & 0 < \alpha < 1, \\
u_t(x,t), & \alpha = 1.
\end{cases}
\end{align}
\tag{1.3}

We want to recover the solution \(f(x) := u(x,0)\). Due to the ill-posedness, some regularization methods must be applied [12-14]. But in this paper, we use a new fractional Tikhonov method.

The outline of the paper is as follows: in Section 2, we give some preliminary results; In order to overcome the difficulty, a novel a priori bound is introduced in Section 2, a conditional stability result is also given; Meanwhile, we propose a fractional Tikhonov regularization method and give convergence estimate under a priori assumption in section 4.

2 Preliminaries

Throughout this paper, we use the following definitions and Lemmas.

**Lemma 2.1.** ([15]) There exist \(\{\lambda_j\}_{j \in \mathbb{N}} \in \mathbb{R}\) and \(\{\phi_j\}_{j \in \mathbb{N}} \in D(A)\) such that
\begin{itemize}
  \item[(i)] \(A(\phi_j) = \lambda_j \phi_j\),
  \item[(ii)] \(0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots\),
  \item[(iii)] \(\lim_{j \to \infty} \lambda_j = \infty\), furthermore, for each \(f \in L^2(\Omega)\),
\end{itemize}
\begin{align}
f = \sum_{j=1}^\infty \langle f, \phi_j \rangle \phi_j \quad \text{in} \quad L^2 \tag{2.1}
\end{align}

and for each \(i,j \in \mathbb{N}\),
\begin{align}
\langle \phi_i, \phi_j \rangle = \begin{cases}
1, & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases} \tag{2.2}
\end{align}

that is, \(\{\phi_j\}_{j \in \mathbb{N}}\) is an orthonormal basis of \(L^2(\Omega)\).