JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **31**, No. 3, pp. 193-213

doi: 10.4208/jpde.v31.n3.1
September 2018

Finite Difference Method for $(2+1)$-Kuramoto-Sivashinsky Equation

BEZIA Abdelhamid[1,*] and MABROUK Anouar Ben[2,3,4]

[1] *Algebra and Number Theory Laboratory, Faculty of Mathematics, University of Sciences and Technology Houari Boumediene, BP 32 EL-Alia 16111, Bab Ezzouar, Algiers, Algeria.*
[2] *Département de Mathématiques, Institut Supérieur de Mathámatiques Appliquées et Informatique de Kairouan, Avenue Assad Ibn Al-Fourat, Kairouan 3100, Tunisia.*
[3] *Laboratory of Algebra, Number Theory and Nonlinear Analysis. Department of Mathematics, Faculty of Sciences, University of Monastir, 5000 Monastir, Tunisia.*
[4] *Department of Mathematics, College of Sciences, Tabuk University, Saudi Arabia Kingdom.*

Received 31 December 2015; Accepted 8 July 2016

Abstract. This paper investigates a solution technique for solving a two-dimensional Kuramoto-Sivashinsky equation discretized using a finite difference method. It consists of an order reduction method into a coupled system of second-order equations, and to formulate the fully discretized, implicit time-marched system as a Lyapunov-Sylvester matrix equation. Convergence and stability is examined using Lyapunov criterion and manipulating generalized Lyapunov-Sylvester operators. Some numerical implementations are provided at the end to validate the theoretical results.

AMS Subject Classifications: 65M06, 65M12, 65M22, 15A30, 37B25.

Key Words: Kuramoto-Sivashinsky equation, Finite difference method, Lyapunov-Sylvester operators.

1 Introduction

Kuramoto-Sivashinsky (KS) equation is one of the well known models for for chaotic spatially extended systems [9]. The KS equation arises in the description of stability of flame fronts, reaction-diffusion systems and many other physical settings [10, 12]. Similarly to [20], in the context of the present paper the two-dimensional generalized KS

*Corresponding author. *Email addresses:* `abdelhamid.bezia@gmail.com` (A. Bezia), `anouar.benmabrouk@fsm.rnu.tn` (A. B. Mabrouk)

equation describes the evolution of a $(2+1)$-dimensional surface defined as a function on a two-dimensional plane and is growing in the direction perpendicular to that plane. Therefore, the present paper is devoted to the development of a computational method based on two-dimensional finite difference scheme to approximate the solution of the nonlinear KS equation

$$\frac{\partial u}{\partial t}=q\Delta u-\kappa\Delta^2 u+\lambda\|\nabla u\|^2, \quad ((x,y),t)\in\Omega\times(t_0,+\infty), \tag{1.1}$$

with initial conditions

$$u(x,y,t_0)=\varphi(x,y); \quad (x,y)\in\Omega \tag{1.2}$$

and boundary conditions

$$\frac{\partial u}{\partial \eta}(x,y,t)=0; \quad ((x,y),t)\in\partial\Omega\times(t_0,+\infty), \tag{1.3}$$

on a rectangular domain $\Omega=[L_0,L_1]\times[L_0,L_1]$ in $\mathbb{R}^2$, $t_0\geq 0$ is a real parameter fixed as the initial time. $\frac{\partial}{\partial t}$ is the time derivative, ∇ is the space gradient operator and $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplace operator in $\mathbb{R}^2$, q,κ,λ are real parameters. φ and ψ are twice differentiable real valued functions on Ω.

We propose to apply an order reduction of the derivation and thus to solve a coupled system of equation involving second order differential operators. We set $v=qu-\kappa\Delta u$ and thus we have to solve the system

$$\begin{cases} \frac{\partial u}{\partial t}=\Delta v+\lambda\|\nabla u\|^2, & (x,y,t)\in\Omega\times(t_0,+\infty) \\ v=qu-\kappa\Delta u, & (x,y,t)\in\Omega\times(t_0,+\infty) \\ (u,v)(x,y,t_0)=(\varphi,\psi)(x,y), & (x,y)\in\Omega \\ \overrightarrow{\nabla}(u,v)(x,y,t)=0, & (x,y,t)\in\partial\Omega\times(t_0,+\infty). \end{cases} \tag{1.4}$$

The Kuramoto-Sivashinsky equation (KS) is one of the most famous equations in math-physics for many decades. It has its origin in the work of Kuramoto since the 70-th decade of the 20-th century in his study of reaction-diffusion equation [21]. The equation was then considered by Sivashinsky in modeling small thermal diffusion instabilities for laminar flames and modeling the reference flux of a film layer on an inclined plane [32, 33]. Since then the KS equation has experienced a growing development in theoretical mathematics, numerical as well as physical mechanics, nonlinear physics, hydrodynamics [28], in combustion theory, chemistry, plasma physics, particle distributions advection, surface morphology, ...etc.

For example, in [1], an anisotropic version of the KS equation has been proposed leading to global resolutions of the equation on rectangular domains. Sufficient conditions were given for the existence of global solution. See for example [6–13].