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Klein-Gordon-Maxwell-Proca Type Systems in the Electro-Magneto-Static Case

HEBEY Emmanuel[1,*] and THIZY Pierre-Damien[2]

[1] *Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France.*
[2] *Institut Camille Jordan, Université Claude Bernard Lyon 1, Bâtiment Braconnier, 21 avenue Claude Bernard, 69622 Villeurbanne Cedex, France.*

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Abstract. We investigate a Klein-Gordon-Maxwell-Proca type system in the context of closed 3-dimensional manifolds. We prove existence of solutions and compactness of the system both in the subcritical and in the critical case.

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1 Introduction

Let (M,g) be a 3-dimensional closed manifold. We look for systems of equations with unknowns (u,v,A), where u and v are functions and A is a 1-form, which express like

$$\begin{cases} \Delta_g u+\Phi(x,v,A)u=u^{p-1}, \\ \Delta_g v+\left(b+q^2u^2\right)v=qu^2, \\ \Delta_g A+bA=q\left(\nabla S-qA\right)u^2, \end{cases} \tag{1.1}$$

where $q>0$, $\Phi(x,v,A)=a-\omega^2(qv-1)^2+|\nabla S-qA|^2$, $\omega\in\mathbb{R}$, $a,b,S\in C^\infty(M)$ are smooth functions with $a,b>0$ in M, $\Delta_g=-\mathrm{div}_g\nabla$ is the Laplace-Beltrami operator when acting on functions u and v, $\Delta_g=\delta d+d\delta$ is the Hodge-de Rham Laplacian when acting on 1-forms A

*Corresponding author. *Email addresses:* `Emmanuel.Hebey@u-cergy.fr` (E. Hebey), `pierre-damien.thizy@univ-lyon1.fr` (P. D. Thizy)

and $p\in(2,6]$ (d is the differential, δ is the codifferential, 6 is the critical Sobolev exponent). Systems of equations like (1.1) are derived from the full KGMP system when we look for solutions of such systems in the form $\Psi(x,t)=u(x,t)e^{iS(x,t)}$ with u depending only on x and S in the splitted form $S(x,t)=S(x)-\omega t$. The full KGMP system (see the discussion in Hebey and Truong [1] and Section 2 below) for solutions like $\Psi(x,t)=u(x,t)e^{iS(x,t)}$ is written as

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}+\Delta_g u+m_0^2 u=u^{p-1}+\left(\left(\dfrac{\partial S}{\partial t}+q\varphi\right)^2-|\nabla S-qA|^2\right)u, \\ \dfrac{\partial}{\partial t}\left(\left(\dfrac{\partial S}{\partial t}+q\varphi\right)u^2\right)-\nabla\cdot\left((\nabla S-qA)u^2\right)=0, \\ -\nabla\cdot\left(\dfrac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2\varphi+q\left(\dfrac{\partial S}{\partial t}+q\varphi\right)u^2=0, \\ \overline{\Delta}_g A+\dfrac{\partial}{\partial t}\left(\dfrac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2 A=q(\nabla S-qA)u^2, \end{cases} \tag{1.2}$$

where $\Delta_g=-\mathrm{div}_g\nabla$ is the Laplace-Beltrami operator, $\overline{\Delta}_g=\delta d$ is half the Laplacian acting on forms, and δ is the codifferential. We assume here that we are in the static case, where $\partial_t u\equiv 0$, $\partial_t A\equiv 0$ and $\partial_t\varphi\equiv 0$. We look for solutions with $S(x,t)=S(x)-\omega t$. Such type of solutions were introduced in the very nice paper by Benci and Fortunato [2] for the Klein-Gordon-Maxwell equations in $\mathbb{R}^3$ (see also D'Avenia, Mederski and Pomponio [3]). A special choice of S in these papers gives rise to vortex solutions of the system. In the case of (1.2), for such solutions, namely for solutions like $\Psi(x,t)=u(x)e^{i(S(x)-\omega t)}$, the full KGMP system (1.2) in the static case writes as

$$\begin{cases} \Delta_g u+m_0^2 u=u^{p-1}+\left((q\varphi-\omega)^2-|\nabla S-qA|^2\right)u, \\ \nabla\cdot\left((\nabla S-qA)u^2\right)=0, \\ \Delta_g\varphi+m_1^2\varphi+q(q\varphi-\omega)u^2=0, \\ \overline{\Delta}_g A+m_1^2 A=q(\nabla S-qA)u^2. \end{cases} \tag{1.3}$$

Thanks to the fourth equation in this system, since $m_1\neq 0$ and $\delta\overline{\Delta}_g=0$, the second equation in (1.3) can be omitted and replaced by the Coulomb gauge equation $\delta A=0$. Also the second equation is automatically satisfied in the Klein-Gordon-Maxwell setting for which $m_1=0$ (since $\delta\overline{\Delta}_g=0$). When $\delta A=0$, we get that $\overline{\Delta}_g A=\Delta_g A$, where $\Delta_g=d\delta+\delta d$ is the Hodge-de Rham Laplacian on forms. Letting $\varphi=\omega v$, and if we replace m_0^2 by a positive function a, and m_1^2 by a positive function b, the system (1.3) reduces to (1.1) when we forget about the Coulomb gauge equation $\delta A=0$. It's not clear that much can be said about (1.1). The goal in this paper is to prove that despite it's intricate structure, strong results can be proved on this system. We recall that a coercive operator like $\Delta_g+\Lambda_g$ has positive mass if the regular part of its Green's function is positive on the diagonal. Our main result is the following theorem. More results are proved in the sequel. The subscript R in the notation C_R^∞ refers to the fact that a,b,S in the theorem are (real-valued) functions.