JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **30**, No. 3, pp. 232-253

doi: 10.4208/jpde.v30.n3.4
September 2017

Solutions to a 3D Burgers Equation with Initial Discontinuity That Are Two Disjoint Spheres

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Received 10 December 2016; Accepted 7 February 2017

Abstract. We study the singular structure of a kind of three dimensional non-selfsimilar global solutions and their interaction for quasilinear hyperbolic conservation laws. The initial discontinuity is two disjoint unit spheres and initial data just contain two different constant states, the global solutions and some new phenomena are discovered. We give the solutions in $0<t<T^*$ and $T^*<t$, and at $t=T^*$, the two basic shock waves and the constant state u_- are disappeared.Then, we find a new shock wave between two rarefaction by R-H condition. Finally, we show the limit of the solution when $t\to\infty$. A technique is proposed to construct the three dimensional shock wave without dimensional reduction or coordinate transformation.

AMS Subject Classifications: 35L65, 35L67

Chinese Library Classifications: O175.27

Key Words: Initial discontinuity; Burgers equation; elementary wave; global solution.

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1 Introduction

The Riemann problem of n-dimensional conservation law has the following form

$$u_t+\sum_{i=1}^{n}\frac{\partial f_i(u)}{\partial x_i}=0, \tag{1.1}$$

with the initial value

$$u(x,t)|_{t=0}=u_0(x)=\begin{cases} u_-, & M(x)<0,\\ u_+, & M(x)>0,\end{cases} \tag{1.2}$$

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where $x=(x_1,\ldots,x_n), u=u(x,t), f_i(u)\in C^2(R), i=1,\ldots,n, M(x)\in C^1(R^n)$, and $M(x)=0$ is a smooth dimensional manifold and divides R^n into two infinite parts.

The Cauchy problem for Eq. (1.1) was proved by Conway and Smoller [1]. Kruzkov [2] and Vol'pert [3] proved weak solution uniquely exists, if it satisfies some kind of entropy condition. Here, (1.1) and (1.2) can be considered as another generalization of one dimensional Riemann problem. In 1-D case, u_- and u_+ disjoint at a point, while a point is with zero dimension, the initial discontinuity can be a curve in 2-D case or be a surface in 3-D case etc. So,the above assumption (1.2) of initial discontinuity is natural in multi-dimensional situation. In [4], the author construct shock wave and rarefaction wave solutions, especially rarefaction wave is got by constructing implicit function, instead of the usual self-similar method.

In this paper, we try to study the singular structure for three dimensional global solutions and interaction of three dimensional elementary wave from a different perspective. Initial discontinuity is a smooth three dimensional sphere and initial value just contain two different constant states. This is a basic problem for investigating structure and development of global solution. Obviously, these kind of solutions are not self-similar and are more general and conventional methods are invalid here. In [4], a new multi-dimensional method was proposed to study such kind of multi-dimensional non-selfsimilar solutions, some fundamental results on non-selfsimilar elementary wave are discovered. The main results of [4] are as follows:

Definition 1.1. *$u(x,t)$ is the weak solution of (1.1)–(1.2) if*

$$\int_{R^+}\int_{R^n}\left(u\frac{\partial\phi}{\partial t}+\sum_{i=1}^{n}f_i(u)\frac{\partial\phi}{\partial x_i}\right)\mathrm{d}x\mathrm{d}t+\int_{t=0}u_0(x)\phi(x,0)\mathrm{d}x=0, \tag{1.3}$$

for all test function $\phi\in C_0^\infty(R^n\times R^+)$.

Definition 1.2. (Condition $H(H')$)*: For all $x\in\{x|M(x)=0,x\in R^n\}$ and u between u_- and u_+, the following form*

$$\sum_{i=1}^{n}M_{x_i}f_i''(u)>0\,(<0), \tag{1.4}$$

holds, we call (1.1)–(1.2) satisfies condition $H(H')$, where $M_{x_i}=M_{x_i}(x)\big|_{x\in\{x|M(x)=0\}}, i=1,\ldots,n$, $u\in(a,b)$, a,b is finite or ∞.

Proposition 1.1. ([4]) *Assume Condition $H(H')$ is satisfied. For any for $\forall x\in\{x|M(x)=0\}$, if $u_->u_+(u_-<u_+)$, u is an n-dimensional shock wave solution, given by*

$$u(x,t)=\begin{cases} u_-, & M\left(x_1-\frac{[f_1]}{[u]}t,x_2-\frac{[f_2]}{[u]}t,\cdots,x_n-\frac{[f_n]}{[u]}t\right)<0,\\ u_+, & M\left(x_1-\frac{[f_1]}{[u]}t,x_2-\frac{[f_2]}{[u]}t,\cdots,x_n-\frac{[f_n]}{[u]}t\right)>0,\end{cases} \tag{1.5}$$